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$\hbar$ -perturbative solutions of quantum Snyder and Yang models with parameters describing spontaneous symmetry breaking

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We introduce the perturbative $\hbar$-power series ($\hbar$ - Planck constant) providing the algebraic solutions of $D = 4$ quantum Snyder and Yang models which describe relativistic quantum space-times and Lorentz-covariant quantum phase spaces. We argue that if in these series the zero order ($\hbar$-independent) terms are non-vanishing they describe the spontaneous symmetry breaking (SSB) parameters of Lie-algebraic symmetries which characterize the considered models ($D = 4$ dS symmetry in Snyder and $D = 5$ dS symmetry in Yang cases). The consecutive terms in $\hbar$-power series can be calculated explicitly if we supplement the SSB order parameters (Nambu-Goldstone or NG modes) by dual set of commutative momenta, which together define the canonical tensorial Heisenberg algebra.

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1. Introduction

It is well known (see e.g. [1–5]) that in quantum theories one can consider two ways of symmetry breaking. The first, explicit symmetry breaking, leads to modified symmetry properties of algebraic structure in considered quantum models (e.g. of action integrals, quantum equations of motion etc.). The second way, describing the case of spontaneous symmetry breaking (SSB), leads to modified symmetries of the solutions and quantum states obtained as the particular SSB realizations of basic algebraic structure. The SSB effects have been considered in Quantum Mechanics (QM) and Quantum Field Theory (QFT) models (see e.g. [6–9]), in particular in the Standard Model (SM), which describes the theory of elementary particles by the tools of QFT (see e.g. [10, 11]). It appears also that in SM the suitable SSB of local gauge symmetries generates, by means of Higgs mechanism [12, 13], the mass parameters which are necessary for the physical applications of SM.

In this paper we consider, in the presence of Quantum Gravity (QG), the effects of the SSB of quantum space-time symmetries, and for such a purpose we use the formalism of noncommutative (NC) geometry, used already since a long time for the description of passage from classical to quantum gravity. It should be observed that using the tools of NC geometry (see e.g. [14–17]), there have been obtained various models (see e.g. [18–22]) describing $D = 4$ quantum space-times, quantum deformed phase spaces and quantum symmetry groups. The NC models which preserved firstly the $D = 4$ relativistic covariance were introduced as early as in 1947 by Snyder and Yang. These models were considered in numerous papers (see e.g. [23–29]), where however the appearance of SSB effects have not been yet studied. It should be recalled, however, that in [27, 30–32] the Snyder type models were solved perturbatively as embedded in the canonical vectorial and tensorial Heisenberg algebras [33, 34]. In this paper, we will show that the introduction of explicit $\hbar$-dependence and the use of perturbation theory described by $\hbar-$power series permits to provide the SSB interpretation of the obtained results [1].

2. Quantum $D = 4$ Snyder algebra, spontaneous symmetry breaking and degenerate vacua

Firstly, we outline briefly the algebraic $\hbar$-dependent description of the Snyder model. The classical relativistic Minkowski space-time coordinates $x_\mu \in \mathbb{M}^{3,1}$ can be introduced as irreducible vectorial representation of the $D = 4$ Lorentz algebra:

$$[M_{\mu\nu}, M_{\rho\tau}] = i(\eta_{\mu\rho}M_{\nu\tau} - \eta_{\mu\tau}M_{\nu\rho} + \eta_{\nu\tau}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\tau}). \tag{1}$$

The relativistic Lorentz covariance of $\mathbb{M}^{3,1}$ can be expressed by the relation:

$$[M_{\mu\nu}, x_\rho] = i(\eta_{\mu\rho}x_\nu - \eta_{\nu\rho}x_\mu). \tag{2}$$

The covariance of fourmomenta $p_\mu \in \mathbb{P}^{3,1}$ is defined in an analogous way:

$$[M_{\mu\nu}, p_\rho] = i(\eta_{\mu\rho}p_\nu - \eta_{\nu\rho}p_\mu). \tag{3}$$

[1] See also [35] where such interpretation of SSB in Snyder and Yang models has been firstly presented; for the discussion of $\hbar$-power expansions in quantum theories, see e.g. [36].

The quantum relativistic space-time coordinates $\hat{x}_\mu \in \hat{\mathbb{X}}^{3,1}$ and quantum fourmomenta $\hat{p}_\mu \in \hat{\mathbb{P}}^{3,1}$ will be specified if we supplement the algebraic noncommutativity relations (1-3) by the set of explicit commutators: $\left[\hat{x}_\mu, \hat{x}_\nu\right]$, $\left[\hat{p}_\mu, \hat{p}_\nu\right]$ and $\left[\hat{x}_\mu, \hat{p}_\nu\right]$. For further study, we should pass from the classical to quantum Lorentz algebra by the following rescaling of the generators:

$$\hat{M}_{\mu\nu} = \hbar M_{\mu\nu} \tag{4}$$

which leads to the following $\hbar$-dependent Lie-algebraic relations:

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i\hbar(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau}). \tag{5}$$

By using (4) we obtain that the covariance relations (2, 3) take the form:

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i\hbar(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu), \tag{6}$$

$$[\hat{M}_{\mu\nu}, \hat{p}_\rho] = i\hbar(\eta_{\mu\rho}\hat{p}_\nu - \eta_{\nu\rho}\hat{p}_\mu). \tag{7}$$

We specify the Snyder quantum space-time by introducing Snyder NC coordinates $\hat{x}_\mu$ satisfying the basic relation

$$[\hat{x}_\mu, \hat{x}_\nu] = il^2 M_{\mu\nu} = i\frac{l^2}{\hbar}\hat{M}_{\mu\nu}, \tag{8}$$

where l is an elementary length. If we assume that $c = 1$, by using the Compton wavelength formula (see e.g. [36]), one can replace l by the inverse of mass M, multiplied by $\hbar$, i.e.

$$l = \frac{\hbar}{Mc} \longrightarrow l = \frac{\hbar}{M} \tag{9}$$

In this way we obtain:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\frac{\hbar}{M^2}\hat{M}_{\mu\nu}. \tag{10}$$

Snyder quantum space-time coordinates, with the Lorentz covariance relation (6) and their NC structure provided by (10), can be described by the curved NC translations spanning the $D = 4$ dS coset $o(4,1)/o(3,1)$ as

$$\hat{M}_{4\mu} = M\hat{x}_\mu, \qquad (\mu = 0, 1, 2, 3). \tag{11}$$

It follows, from relations (5), (6), (11) that the generators $\hat{M}_{AB}$ $(A, B = 0, 1, 2, 3, 4)$ providing the quantum $D = 4$ dS algebra

$$[\hat{M}_{AB}, \hat{M}_{CD}] = i\hbar(\eta_{AC}\hat{M}_{BD} - \eta_{AD}\hat{M}_{BC} + \eta_{BD}\hat{M}_{AC} - \eta_{BC}\hat{M}_{AD}). \tag{12}$$

describe the quantum $D = 4$ Snyder model.

We recall that the difference between the symmetries of equations of motion and the restricted symmetries of explicit solutions in the model lead to the appearance of SSB. Namely, the SSB arises in the model when the symmetries describing covariance of the algebraic structure are reduced due to the presence of numerical order parameters in explicit solutions, as well as in background fields (see e.g. [37]). In our case the algebra (12) is $D = 4$ Lorentz and $D = 4$ dS covariant, but the realizations of generators $\hat{M}_{AB}$ may contain parts which violate both the Lorentz and dS covariance. In the general case, one can decompose $\hat{M}_{AB}$, into their classical and quantum parts, as follows:

$$\hat{M}_{AB} = M^{cl}_{AB} + \hat{M}^q_{AB} = M^{cl}_{AB} + \hbar M^q_{AB} \tag{13}$$

where the classical Abelian part is given by the 5 × 5 antisymmetric matrices

$$M^{cl}_{AB} = \lim_{\hbar\to 0} \hat{M}_{AB} = x_{AB} = -x_{BA}. \tag{14}$$

In such a way, we introduce the NG degrees of freedom and the spontaneous symmetry breaking of the quantum $D = 4$ dS algebra (12)[2]. We see that, if we describe $\hat{M}_{AB}$ in (13) as the operator-valued $\hbar$-power series, the zero order terms in $\hbar$-expansions provide the classical parameters x_{AB}. In particular, following the construction of the so-called "phenomenological Lagrangians" [8, 9], one can distinguish two separate cases:

i) if $x_{\mu\nu} = 0$ and $x_{4\mu} = Mx_\mu \neq 0$ one obtains the Snyder models with preserved linear Lorentz covariance.

ii) in general case one can assume that $x_\mu \neq 0$ as well as $x_{\mu\nu} \neq 0$, what may lead to the spontaneous symmetry breaking of any part of the $D = 4$ dS algebra $o(4, 1)$[3].

If we introduce the vacuum state $|0\rangle$ with the lowest value of energy one can describe equivalently the relations (13)-(14) as follows:

$$M^{cl}_{AB}|0\rangle = x_{AB}|0\rangle, \qquad \hat{M}^{q}_{AB}|0\rangle = 0 \tag{15}$$

Because the parameters x_{AB} are $\mathbb{C}$-numbers, one obtains that

$$x_{AB} = \langle 0|\hat{M}_{AB}|0\rangle = (x_{\mu\nu} = -x_{\nu\mu},\, x_{4\mu} = -x_{\mu 4}) \tag{16}$$

and $x_{4\mu} = Mx_\mu$.

In the presence of SSB the vacuum (lowest energy state of a system) is not unique and we obtain the set of degenerate vacua, which may depend on the parameters x_{AB}. In such a case part of the symmetries providing the covariance of the Snyder model do not leave all vacua states invariant, what leads to the excitation of massless Goldstone bosons [7]. It should be added that in order to specify the set of SSB vacua one should employ the notion of a Hamiltonian, which can be calculated e.g. when Snyder algebra is derived from the Lagrangian formulation of the Snyder particle model (see e.g. [38, 39]).

3. $\hbar$-perturbative solutions of spontaneously broken $D = 4$ Snyder models

Our aim in this paper is to employ the perturbative $\hbar$-expansions in order to consider the solutions of Snyder model with spontaneously broken $D = 4$ dS symmetries. We expand the $o(4, 1)$ generators $\hat{M}_{AB}$ in the $\hbar$-power series

$$\hat{M}_{AB} = M^{(0)}_{AB} + \hbar\hat{M}^{(1)}_{AB} + \hbar^2\hat{M}^{(2)}_{AB} + \dots \tag{17}$$

[2]The non-vanishing classical part M^{cl}_{AB} may define as well the classical background parameters which lead to spontaneously broken symmetries.

[3]If we put $G = o(4, 1)$, the coset decomposition $G = H \otimes \frac{G}{H}$ provides the construction of H-covariant phenomenological Lagrangians. If all symmetries of G are spontaneously broken, one should put $H = \mathbf{1}$ and introduce in formula (13) 10 non-vanishing parameters x_{AB} (see 14).

where $M_{AB}^{(0)} = M_{AB}^{cl} = x_{AB}$ describe the non-vanishing classical $\hbar$-independent part of the quantum generators $\hat{M}_{AB}$, what indicates the presence of SSB. The variables x_{AB} (see (16)) can be treated as the Nambu-Goldstone (NG) coordinates which can be extended by adding dual NG momenta $p_{AB} = -p_{BA}$ and subsequently form the quantum phase space (x_{AB}, p_{AB}) satisfying the tensorial canonical Heisenberg algebra[4]

$$[x_{AB}, x_{CD}] = [p_{AB}, p_{CD}] = 0, \qquad [x_{AB}, p_{CD}] = i\hbar(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}). \tag{18}$$

In order to calculate the $\hbar$-perturbative solutions of $D = 4$ Snyder model, the presence of non-vanishing tensorial momenta is needed for the iterative explicit calculations of consecutive terms $\hat{M}_{AB}^{(n)}$ $(n = 1, 2, \ldots)$ in $\hbar$-power series (17). Such tensorial Heisenberg algebra (18) has been already postulated in earlier papers (see e.g. [33, 34]) but without exposing the relation to SSB and the perturbative solutions as $\hbar$-power series. It follows that x_{AB} describe the Abelian order parameters which parametrize the set of possible spontaneously broken degenerate vacuum states $|0; x_{AB}\rangle$ if they satisfy the relations $\hat{M}_{AB}|0; x_{AB}\rangle = x_{AB}|0; x_{AB}\rangle$ and have the same minimal energy (Hamiltonian) eigenvalue.

α) perturbative $\hbar$-expansion: first order in $\hbar$

From relation (12) one gets:

$$\left[x_{AB}, \hat{M}_{CD}{}^{(1)}\right] + \left[\hat{M}_{AB}{}^{(1)}, x_{CD}\right] = i(\eta_{AC}x_{BD} - \eta_{AD}x_{BC} - \eta_{BC}x_{AD} + \eta_{BD}x_{AC}) \tag{19}$$

where $\eta_{AB} = diag(-1, 1, 1, 1, 1)$ and relations (2,10) lead to

$$\left[x_{\mu\nu}, \hat{M}_{\rho\sigma}{}^{(1)}\right] - \left[x_{\rho\sigma}, \hat{M}_{\mu\nu}{}^{(1)}\right] = i(\eta_{\mu\rho}x_{\nu\sigma} - \eta_{\mu\sigma}x_{\nu\rho} - \eta_{\nu\rho}x_{\mu\sigma} + \eta_{\nu\sigma}x_{\mu\rho}), \tag{20}$$

$$\left[x_{\mu}, \hat{M}_{\rho\sigma}{}^{(1)}\right] - \left[x_{\rho\sigma}, \hat{x}_{\mu}{}^{(1)}\right] = i(\eta_{\mu\sigma}x_{\rho} - \eta_{\mu\rho}x_{\sigma}), \tag{21}$$

$$\left[x_{\mu}, \hat{x}_{\nu}{}^{(1)}\right] - \left[x_{\nu}, \hat{x}_{\mu}{}^{(1)}\right] = \frac{i}{M^2}x_{\mu\nu}. \tag{22}$$

In order to solve the relations (19-22) we employ the generalized momenta $p_{AB} = \left(p_{\mu\nu}, p_{\mu}\right)$ (see (18)). From (20) and (21) one can obtain a particular solution, given by[5]

$$\hbar\hat{M}_{\mu\nu;S}^{(1)} = \frac{1}{2}\left(x_{\mu}{}^{\rho}p_{\nu\rho} - x_{\nu}{}^{\rho}p_{\mu\rho}\right) + x_{\mu}p_{\nu} - x_{\nu}p_{\mu} \tag{23}$$

and in consistency with (22)

$$\hbar\hat{x}_{\mu;S}^{(1)} = -\frac{1}{2M^2}x_{\mu\rho}p^{\rho}. \tag{24}$$

The general first order solution depends on one free parameter and can be obtained by a suitable choice of similarity transformations of the particular solutions (23,24).

[4]See also [40], where the NG momenta for Lie algebras were considered.

[5]Subindex S denotes the Snyder case. In (23,24) the factor $\hbar$ on the left hand side reflects the proportionality of quantum-mechanical momenta to $\hbar$.

β) perturbative $\hbar$-expansion: second order in $\hbar$

The second order counterpart of relation (19) looks as follows:

$$\left[x_{AB}, \hat{M}^{(2)}_{CD}\right]-\left[x_{CD}, \hat{M}^{(2)}_{AB}\right]+\left[\hat{M}^{(1)}_{AB}, \hat{M}^{(1)}_{CD}\right] = i(\eta_{AC}\hat{M}^{(1)}_{BD}+\eta_{BD}\hat{M}^{(1)}_{AC}-\eta_{BC}\hat{M}^{(1)}_{AD}-\eta_{AD}\hat{M}^{(1)}_{BC}) \quad (25)$$

which leads to:

$$\left[x_{\mu\nu}, \hat{M}^{(2)}_{\rho\sigma}\right] - \left[x_{\rho\sigma}, \hat{M}^{(2)}_{\mu\nu}\right] = i(\eta_{\mu\rho}\hat{M}^{(1)}_{\nu\sigma}+\eta_{\nu\sigma}\hat{M}^{(1)}_{\mu\rho} - \eta_{\nu\rho}\hat{M}^{(1)}_{\mu\sigma} - \eta_{\mu\sigma}\hat{M}^{(1)}_{\nu\rho}) - \left[\hat{M}^{(1)}_{\mu\nu}, \hat{M}^{(1)}_{\rho\sigma}\right], \quad (26)$$

$$\left[x_{\mu}, \hat{M}^{(2)}_{\rho\sigma}\right] - \left[x_{\rho\sigma}, \hat{x}^{(2)}_{\mu}\right] = i(\eta_{\mu\sigma}\hat{x}^{(1)}_{\rho} - \eta_{\mu\rho}\hat{x}^{(1)}_{\sigma}) - \left[\hat{x}^{(1)}_{\mu}, \hat{M}^{(1)}_{\rho\sigma}\right], \quad (27)$$

$$\left[x_{\mu}, \hat{x}^{(2)}_{\sigma}\right] - \left[x_{\sigma}, \hat{x}^{(2)}_{\mu}\right] = \frac{i}{M^2}\hat{M}^{(1)}_{\mu\sigma} - \left[\hat{x}^{(1)}_{\mu}, \hat{x}^{(1)}_{\sigma}\right]. \quad (28)$$

Substituting in (26-28) the solutions (23,24) one gets, in the second order of $\hbar$, the following particular solution:

$$\hbar^2\hat{M}^{(2)}_{\mu\nu;S} = -\frac{1}{12}\left(x_{\mu\rho}p^{\rho\sigma}p_{\nu\sigma} - x_{\nu\rho}p^{\rho\sigma}p_{\mu\sigma} - 2x^{\rho\sigma}p_{\mu\rho}p_{\nu\sigma}\right), \quad (29)$$

$$\hbar^2\hat{x}^{(2)}_{\mu;S} = \frac{1}{M^2}\left(x_{\rho}p^{\rho}p_{\mu} + \frac{1}{4}\left(x_{\mu\rho}p_{\rho\sigma}p_{\sigma} + x^{\rho\sigma}p_{\rho}p_{\mu\sigma}\right)\right). \quad (30)$$

General parameter-dependent solutions in the second $\hbar$-order can be obtained from the formulae (29,30) by performing suitable similarity transformations. One can also show that, in the perturbative n-th order of $\hbar$, the solutions $(\hat{x}^{(n)}_{\mu;S}, \hat{M}^{(n)}_{\mu\nu;S})$ are n-linear in momenta $p_{AB} = (p_{\mu\nu}, p_{\mu})$.

4. $\hbar$-perturbative solutions of spontaneously broken $D = 4$ Yang models

$D = 4$ Yang model (see e.g. [19], [41]-[43]) is described by fifteen generators of $D = 5$ dS algebra $o(5,1)$ $(K, L = 0,1,2,3,4,5)$

$$\hat{M}_{KL} = \left(\hat{M}_{\mu\nu}, \hat{M}_{4\mu} = M\hat{x}_{\mu}, \hat{M}_{5\mu} = R\hat{q}_{\mu}, \hat{M}_{45} = MR\hat{r}\right) \quad (31)$$

which provides $D = 4$ Lorentz-covariant quantum-deformed relativistic Heisenberg algebra with two deformation parameters (M, R) (of length dimensions $[M] = L^{-1}$, $[R] = L$) and dimensionless scalar Abelian generator $\hat{r}$, describing internal $o(2)$ symmetries. In general case, in Yang model one can introduce the following fifteen Abelian NG modes, which may break spontaneously all $o(5,1)$ symmetries $(x_{KL} = -x_{LK})$

$$x_{KL} = \left(x_{\mu\nu}, Mx_{\mu}, Rq_{\mu}, MRr\right). \quad (32)$$

To solve the Yang model by using perturbative $\hbar$-expansion one should also introduce fifteen canonically conjugated NG momenta

$$p_{KL} = \left(p_{\mu\nu}, p_{\mu}, k_{\mu}, s\right) \quad (33)$$

satisfying the canonical commutation relations (18), with the additional relations

$$\left[q_{\mu}, k_{\nu}\right] = i\hbar\eta_{\mu\nu}, \qquad [r, s] = i\hbar. \quad (34)$$

Using the variables (32), (33) one can obtain the $\hbar$-perturbative solutions of Yang model, which has been proposed in [19] as the Lie-algebraic extension of the Snyder model [18]. By adding to $o(4,1)$ a fifth space coordinate, one gets the set of Lorentz-covariant formulae (31) where the generators $(\hat{x}_\mu, \hat{q}_\mu, \hat{r})$ describe the quantum-deformed $D=4$ Heisenberg algebra, with one additional Abelian internal symmetry generator.

In Yang model one extends the Snyder relations (1), (2), (10) by the following set of algebraic equations

$$[\hat{q}_\mu, \hat{q}_\nu] = i\frac{\hbar}{R^2}\hat{M}_{\mu\nu}, \tag{35}$$

$$[\hat{x}_\mu, \hat{q}_\nu] = i\frac{\hbar}{MR}\eta_{\mu\nu}\hat{r} \tag{36}$$

$$[\hat{M}_{\mu\nu}, \hat{q}_\rho] = i\hbar\left(\eta_{\nu\rho}\hat{q}_\mu - \eta_{\mu\rho}\hat{q}_\nu\right), \tag{37}$$

$$[\hat{r}, \hat{x}_\mu] = \frac{i\hbar}{M^2}\hat{q}_\mu, \tag{38}$$

$$[\hat{r}, \hat{q}_\mu] = -\frac{i\hbar}{R^2}\hat{x}_\mu \tag{39}$$

where $\hat{q}_\mu = q_\mu^{(cl)} + \hat{q}_\mu^{(q)}$, $\hat{r} = r^{(cl)} + \hat{r}^{(q)}$. The first order $\hbar$-approximation of the algebraic solutions of Yang model is obtained when in the relations (1), (2), (10) and (35-39) we consider only the linear $\hbar$-terms. Besides (20-22) we get $\left(r \equiv \hat{r}^{(0)} = r^{(cl)}\right)$

$$\left[q_\mu, \hat{q}_\nu{}^{(1)}\right] - \left[q_\nu, \hat{q}_\mu{}^{(1)}\right] = \frac{i}{R^2}x_{\mu\nu}, \tag{40}$$

$$\left[x_\mu, \hat{q}_\nu{}^{(1)}\right] - \left[q_\nu, \hat{x}_\mu{}^{(1)}\right] = ir\eta_{\mu\nu}, \tag{41}$$

$$\left[x_{\mu\nu}, \hat{q}_\rho{}^{(1)}\right] + \left[\hat{M}_{\mu\nu}{}^{(1)}, q_\rho\right] = i\left(\eta_{\nu\rho}q_\mu - \eta_{\mu\rho}q_\nu\right), \tag{42}$$

$$\left[r, \hat{x}_\mu{}^{(1)}\right] + \left[\hat{r}^{(1)}, x_\mu\right] = \frac{i}{M^2}q_\mu, \tag{43}$$

$$\left[r, \hat{q}_\mu{}^{(1)}\right] + \left[\hat{r}^{(1)}, q_\mu\right] = -\frac{i}{R^2}x_\mu. \tag{44}$$

For the extended Snyder model, in the first order, we obtained the formulas (23), (24). In Yang model, due to the presence of additional coordinates (q_μ, r) one should add their dual momenta (k_μ, s)) (see (32),(33)) and extend the formulae (23), (24) by terms which are linear in momenta (k_μ, s) (see (33)) as follows:

$$\hbar\hat{M}^{(1)}_{\mu\nu;Y} = \frac{1}{2}\left(x_\mu{}^\rho p_{\nu\rho} - x_\nu{}^\rho p_{\mu\rho}\right) + x_\mu p_\nu - x_\nu p_\mu - q_\mu k_\nu + q_\nu k_\mu, \tag{45}$$

$$\hbar\hat{x}^{(1)}_{\mu;Y} = -\frac{1}{2M^2}x_{\mu\rho}p^\rho + a x_{\mu\rho}k^\rho + b r k_\mu + c q_\mu s, \tag{46}$$

where a, b, c are numerical constants. We add further the formulae:

$$\hbar\hat{q}_\mu{}^{(1)} = -\frac{1}{2R^2}x_{\mu\rho}k^\rho + \tilde{a}x_{\mu\rho}p^\rho + \tilde{b}rp_\mu + \tilde{c}x_\mu s, \tag{47}$$

$$\hbar\hat{r}^{(1)} = dq^\rho p_\rho + f x^\rho k_\rho, \tag{48}$$

introducing additional five numerical constants $\tilde{a}, \tilde{b}, \tilde{c}, d$ and f. The equations (40)-(44) impose the following constraints on eight parameters in (46)-(48):

$$a + \tilde{a} = 0, \qquad \tilde{b} = b + 1, \qquad c - d = \frac{1}{M^2}, \qquad \tilde{c} - f = -\frac{1}{R^2} \tag{49}$$

and imply the absence in formulae (46)-(48) of the terms proportional to $p_{\mu\nu}$. We see therefore that the solutions of equations (40)-(44) linear in $\hbar$ contain four unconstrained numerical parameters a, b, c, f.

The above calculation can be extended to higher orders in $\hbar$. In particular, in the second order of $\hbar$ we get the following set of algebraic equations (besides (26),(27),(28)):

$$\left[q_\nu, \hat{q}_\sigma{}^{(2)}\right] - \left[q_\sigma, \hat{q}_\nu{}^{(2)}\right] = \frac{i}{R^2}\hat{M}^{(1)}_{\nu\sigma} - \left[\hat{q}_\nu{}^{(1)}, \hat{q}_\sigma{}^{(1)}\right], \tag{50}$$

$$\left[x_\nu, \hat{q}_\sigma{}^{(2)}\right] - \left[q_\sigma, \hat{x}_\nu{}^{(2)}\right] = i\eta_{\nu\sigma}\hat{r}^{(1)} - \left[\hat{x}_\nu{}^{(1)}, \hat{q}_\sigma{}^{(1)}\right], \tag{51}$$

$$\left[x_{\mu\nu}, \hat{q}_\sigma{}^{(2)}\right] + \left[\hat{M}_{\mu\nu}{}^{(2)}, q_\sigma\right] = i(\eta_{\mu\sigma}\hat{q}^{(1)}_\nu - \eta_{\nu\sigma}\hat{q}^{(1)}_\mu) - \left[\hat{M}_{\mu\nu}{}^{(1)}, \hat{q}_\sigma{}^{(1)}\right], \tag{52}$$

$$\left[r, \hat{x}_\sigma{}^{(2)}\right] + \left[\hat{r}^{(2)}, x_\sigma\right] = \frac{i}{M^2}\hat{q}^{(1)}_\sigma - \left[\hat{r}^{(1)}, \hat{x}_\sigma{}^{(1)}\right], \tag{53}$$

$$\left[r, \hat{q}_\sigma{}^{(2)}\right] + \left[\hat{r}^{(2)}, q_\sigma\right] = -\frac{i}{R^2}\hat{x}^{(1)}_\sigma - \left[\hat{r}^{(1)}, \hat{q}_\sigma{}^{(1)}\right]. \tag{54}$$

where the second order terms $\hat{x}^{(2)}_\mu, \hat{q}^{(2)}_\mu, \hat{M}^{(2)}_{\mu\nu}, \hat{r}^{(2)}$ extending relations (29,30) are bilinear in momenta variables p_{KL} (see (33)).

5. Concluding remarks

The main idea in the present paper is to show that in the algebraic models describing NC quantum space-times and quantum phase spaces, the effects of SSB can also be present.

The non-commutative quantum space-times and quantum-deformed Heisenberg algebra representing NC phase spaces, both considered as the tools for the description of quantum gravity, are naturally expressed in the form of NC algebras, with the use of the framework of NC geometry. We considered here simple and quite popular Snyder and Yang models, algebraically described by $o(4,1)$ and $o(5,1)$ Lie algebras, which have been often used in the current quantum gravity research, in particular exploiting NC quantum geometry (see [17, 21, 44]). To be more specific, $o(4,1)$ describes an extended Snyder model, in which Snyder quantum space-time is a subspace of a larger non-commutative algebra, which includes Lorentz symmetry generators. The algebraic basis of the extended Snyder algebra is spanned by the generators defining Snyder quantum space-time coordinates $\hat{x}_\mu = \frac{1}{M}\hat{x}_{4\mu}$ as well as quantum tensorial coordinates $\hat{x}_{\mu\nu} = \hat{M}_{\mu\nu}$ describing the Lorentz algebra. Such models have been investigated mostly from the mathematical point of view by several authors (see e.g. [26, 30, 32, 33]), however the interpretation of the new Abelian tensorial coordinates (16) remained unclear, until the recent proposal [35] of a link between the Abelian tensorial coordinates and the modes providing SSB of Lorentz and de Sitter symmetries.

In order to distinguish, in considered quantum algebraic models, the terms which provide SSB we looked for the perturbative solutions expressed as the $\hbar$-power series ($\hbar$ is the Planck constant) for singling out the zero order $\hbar$-independent terms, which describe the classical parts of quantum solutions necessary for the appearance of SSB effects. We add that above construction can be applied in supersymmetrized quantum Snyder and Yang models (see e.g. [45, 46]), with SSB introducing fermionic Grassmanian NG degrees of freedom.

Finally, one can mention that the role of spontaneously broken symmetries, background fields and classical modes in current quantum gravity research and the description of quantum Universe remains, at present, still an open issue.

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