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A quantum reference frame size-accuracy trade-off for quantum channels

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Abstract. The imposition of symmetry upon the nature and structure of quantum observables has recently been extensively studied, with quantum reference frames playing a crucial role. In this paper, we extend this work to quantum transformations, giving quantitative results showing, in direct analogy to the case of observables, that a "large" reference frame is required for non-covariant channels to be well approximated by covariant ones. We apply our findings to the concrete setting of $SU(2)$ symmetry.

Dedicated to the memory of our friend Paul Busch

1. Introduction

The role played by symmetry in the understanding and development of physics can hardly be overstated. It is an important part of the process of building mathematical models of the physical world, and is often crucial for their solubility in both classical and quantum settings. Less widely recognised, however, is the effect that symmetry has in limiting what is measurable. In fact, as is common in gauge theories, e.g., [1], and postulated in, e.g., [2, 3] in the context of quantum reference frames, theoretical quantities which do not commute with a symmetry action are unobservable, even in principle. The tension between this apparent unobservability and the use of such quantities in the description of real physical systems is relieved by noting that any system of interest is a part of a larger whole - there is another system (called the environment, ancilla, reservoir, reference frame, or apparatus) whose presence is often assumed only implicitly and which does not appear in the formulation. It is then possible that unobservable quantities of the system of interest may be re-interpreted as representatives of observable *relative* quantities of system-plus-environment.

In this paper we study the extent to which the same kind of restrictions (due to symmetry) hold in a dynamical context. To make the problem concrete, we must identify, in analogy to the invariance requirement for observables, the right notion of restriction for channels, since it is not unique. This will be discussed in detail in a future publication. One natural example arises in the presence of conserved quantities. The Wigner-Araki-Yanase (WAY) theorem [4–9] shows that conservation laws restrict the accuracy and repeatability properties of a class of quantum measurements. Also in the presence of an additive conserved quantity, Åberg [10] has shown that any unitary dynamics can be approximately realized by preparing "coherent" (thus asymmetric) states of the environment. We here emphasize the necessity of the highly asymmetric (coherent) state (cf. [2, 3, 11]), which requires a large environment in a sense to be described. If the environment is not large, states cannot have enough coherence and the possible dynamics is restricted. In addition, if the symmetry is not Abelian, there may be some restriction due to the uncertainty relation, because large coherence with respect to some observable implies small coherence with respect to its conjugate.

Another possible symmetry constraint is covariance of the dynamics, whose relevance to the reference frame context will be discussed briefly in the next section. As will be shown later, this constraint is weaker than the one given by the presence of conserved quantities. In this paper, we study the possible dynamics of the system under the symmetry constraint on the whole system (object system plus reference). We consider a quantum channel on the system and study how well this (target) channel is approximately realized by a covariant channel on the whole system, contingent upon a choice of state of the environment. We derive a quantitative relation which shows that for the covariant channel to be well approximated by the target channel, high asymmetry/coherence is required for the state of the environment. As an example we apply our relation to a qubit system under $SU(2)$ symmetry.

Such size-versus-inaccuracy trade-offs are already present in the literature in various different contexts (see, e.g., [11–13]), and our findings are broadly in line with other findings - that good accuracy needs large size. Specific mention must be given to [14–16], which has already investigated the dynamical setting and some elegant bounds have been provided, particularly in the unitary case. However, we provide a novel quantitative bound in the dynamical context.

From a technical point of view, the present paper may be regarded as a descendant of [11], in which we derived a quantitative bound in approximating an arbitrary effect by a globally invariant effect. There we reinterpreted the issue as an approximate joint measurement problem of observables and employed a quantitative uncertainty relation [17]. In this paper, we show that a similar technique, which we may call uncertainty relation based method, can be applied also to the approximating channel problem.

2. Symmetry constraints on channels

The principle of symmetry limits the possible observables to invariant ones (see, e.g., [3]). There are some different generalizations of this constraint to channels.

Suppose that we have a system described by a Hilbert space $\mathcal{H}$. By $\mathbf{B}(\mathcal{H})$ we denote the algebra of bounded operators on $\mathcal{H}$. Throughout this paper every Hilbert space we encounter will be finite dimensional. We assume that a finite dimensional connected Lie group G is acting on $\mathcal{H}$. G is assumed to define a true smooth unitary representation on each system, denoted by $U(g)$.

One of the possible constraints on dynamics is given by a conservation law. Suppose that there exists a conserved charge N, which generates a $U(1)$ action. In the situation that the system is closed/isolated, the possible dynamics Λ must satisfy $\Lambda(N) = N$ (in the Heisenberg picture). We thus arrive at the following definition.

Definition 1. Invariant channels *are defined as those* $\Phi : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$ *for which* $\Phi(U(g)) = U(g)$ *for all* $g \in G$*, where* U *is an* $\mathcal{H}$*-representation of* G*.*

On the other hand, suppose that we have a system and are asked to rotate the state around the z-axis by some angle θ. The desired channel is $\Phi(X) = e^{iS_z\theta} X e^{-iS_z\theta}$, where S_z is the z-component of angular momentum. (We work in units in which $\hbar = 1$.) This channel implicitly assumes the existence of a reference frame/system which specifies the z-axis. If we employ another reference frame, the z-component of angular momentum is represented as $U(R)S_zU(R)^*$ with some $R \in SO(3)$ and the corresponding channel becomes $\Phi_R(X) = U(R)\Phi(U(R)^*XU(R))U(R)^*$. If we are not informed which reference frame is to be

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used, we may choose one randomly, in which case the channel is described as

$$\overline{\Phi}(X) = \int \mu(dR)\Phi_R(X), \tag{1}$$

where $\mu(\cdot)$ is (the) Haar measure on $SO(3)$. $\overline{\Phi}$ is an example of a *covariant channel* which we now define.

Definition 2. *A channel* $\Lambda : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$ *is called covariant if and only if*

$$\Lambda(U(g)^* X U(g)) = U(g)^* \Lambda(X) U(g) \tag{2}$$

holds for all $X \in \mathbf{B}(\mathcal{H})$ *and for all* $g \in G$.

As the next proposition shows, invariant channels form an important subclass of covariant channels.

Proposition 1. *Invariant channels are covariant.*

Proof. We begin by using a channel Λ to define an "operator-valued inner product" $\langle\langle A|B\rangle\rangle := \Lambda(A^*B) - \Lambda(A)^*\Lambda(B)$, which satisfies a Cauchy-Schwarz type inequality (see [18] and Lemma 3 in [11]):

$$\|\langle\langle A|B\rangle\rangle\|^2 \leq \|\langle\langle A|A\rangle\rangle\|\|\langle\langle B|B\rangle\rangle\|, \tag{3}$$

where $||\cdot||$ denotes the standard operator norm in $\mathbf{B}(\mathcal{H})$. Suppose that a unitary U is a fixed point, i.e., $\Lambda(U) = U$. Then it holds that

$$\langle\langle U|U\rangle\rangle = \mathbb{1} - \Lambda(U)^*\Lambda(U) = 0.$$

Thus for such a U and arbitrary A we find

$$\langle\langle A|U\rangle\rangle = \Lambda(A^*U) - \Lambda(A^*)\Lambda(U) = \Lambda(A^*U) - \Lambda(A)^*U = 0, \text{ by (3)}.$$

Now Let $\Lambda : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$ be an invariant channel. Then for all $g \in G$,

$$\Lambda(AU(g)) = \Lambda(A)\Lambda(U(g)) = \Lambda(A)U(g).$$

Similarly $\Lambda(U(g)^*B) = U(g)^*\Lambda(B)$, and thus $\Lambda(U(g)^*AU(g)) = U(g)^*\Lambda(A)U(g)$. □

We note that there exist covariant channels which are not invariant; for instance, for any invariant state ω_0, the channel $\Lambda(X) = \omega_0(X)\mathbb{1}$ is covariant but not invariant. However, covariance and invariance are equivalent for unitary channels:

Proposition 2. *Unitary covariant channels are invariant.*

Proof. Let us consider a unitary (and thus automorphic), covariant channel $\Lambda : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$, i.e., $\Lambda(X) = V^*XV$ and $U(g)^*\Lambda(X)U(g) = \Lambda(U(g)^*XU(g))$. Then it holds that for all X

$$\Lambda(U(g)^*XU(g)) = V^*U(g)^*VV^*XVV^*U(g)V = U(g)^*(V^*XV)U(g).$$

Now putting $V^*XV = Z$, we apply $U(g) \cdot U(g)^*$ to the above equation to obtain

$$(U(g)V^*U(g)^*V)Z(V^*U(g)VU(g)^*) = Z,$$

which implies $V^*U(g)VU(g)^* = c(g)\mathbb{1}$ with $|c(g)| = 1$ and $V^*U(g)V = c(g)U(g)$. As the left-hand side satisfies $V^*U(g)VV^*U(g')V = V^*U(gg')V$, $c(g)c(g') = c(gg')$ holds for all $g, g \in G$. Now in a neighborhood of $e \in G$, for each element l of Lie algebra the corresponding generator L exists satisfying $U(e^{ls}) = e^{iLs}$ for sufficiently small $|s|$. If we put the generator of $V^*U(e^{ls})V$ as L', it satisfies $V^*LV = L'$. It in addition satisfies $L' = L + k\mathbb{1}$ for some $k \in \mathbf{R}$ as $V^*U(e^{ls})V = c(e^{ls})U(e^{ls})$ must hold. But as L is bounded (as $\mathcal{H}$ is finite dimensional) and $\|L\| = \|L'\|$ holds, $k = 0$ is the only possible choice. Thus we have shown that for a neighbourhood N_e of $e \in G$ $V^*U(g)V = U(g)$ is satisfied. As G is connected, it is generated by N_e. It implies that $c(g) = 1$ for all $g \in G$. □

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3. The setting and results

As we have seen in the last section, we cannot implement (for instance) the rotation around the z-axis without using a "correct" reference frame. More precisely, we may implement the right rotation but this occurs only by chance. The averaged channel is a covariant $\overline{\Phi}$ which is different from the desired rotation. In the worst case, the discrepancy is larger than the averaged case. Thus we must have a reference frame. Since a reference frame is also a physical system, there should be a quantum description. Our question is then to ask what is the condition on the quantum reference frame so that it works well to implement the desired channel. In the following we formulate the problem in a general setting.

Let G be a connected Lie group. We have a system and a reference frame described by (as always, finite dimensional) Hilbert spaces $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{R}}$, and on each space, G has a smooth true unitary representation $U_{\mathcal{S}}(g)$ and $U_{\mathcal{R}}(g)$. Their composition is written as $U(g) = U_{\mathcal{S}}(g) \otimes U_{\mathcal{R}}(g)$ which acts on $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$. Our purpose is to study how well a general channel $\Lambda : \mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$ is approximately realized by the restriction of a covariant channel $\Phi : \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. We view Φ as representing the "true" transformation, with its restriction representing the transformation with the additional system suppressed. Therefore, Φ satisfies

$$\Phi(U(g)^* X U(g)) = U(g)\Phi(X)U(g)$$

for all $g \in G$ and $X \in \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. On the level of observables, the restriction to the system $\Gamma_{\rho_{\mathcal{R}}} : \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$ is determined by a state $\rho_{\mathcal{R}}$ on $\mathbf{B}(\mathcal{H}_{\mathcal{R}})$ and is defined by the completely positive conditional expectation

$$\mathrm{tr}[\rho_{\mathcal{S}} \Gamma_{\rho_{\mathcal{R}}}(X)] = \mathrm{tr}[(\rho_{\mathcal{S}} \otimes \rho_{\mathcal{R}})X],$$

which holds for all states $\rho_{\mathcal{S}}$ of the system and $X \in \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. In order to define the restriction for channels, we use the natural inclusion $\iota : \mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$, given as

$$\iota(A) = A \otimes \mathbb{1}_{\mathcal{R}}.$$

Then the realized channel is written as $\Phi_{\rho_{\mathcal{R}}} := \Gamma_{\rho_{\mathcal{R}}} \circ \Phi \circ \iota : \mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$, and we wish to quantify the discrepancy between $\Phi_{\rho_{\mathcal{R}}}$ and Λ. As a quantity to characterize the discrepancy, one may employ the norm difference between two channels defined by

$$\|\Phi_{\rho_{\mathcal{R}}} - \Lambda\|_{\mathrm{channel}} := \sup_{X \in \mathbf{B}(\mathcal{H}_{\mathcal{S}}), \|X\|=1} \|\Phi_{\rho_{\mathcal{R}}}(X) - \Lambda(X)\|.$$

For each element of the Lie algebra $\mathfrak{g}$ of a Lie group G there exists a corresponding self-adjoint operator (the generator) acting in $\mathcal{H}_{\mathcal{S}}$. For each $l \in \mathfrak{g}$, there exist operators $L_{\mathcal{S}}$ and $L_{\mathcal{R}}$ satisfying $U_{\mathcal{S}}(e^{ls}) = e^{iL_{\mathcal{S}}s}$ and $U_{\mathcal{R}}(e^{ls}) = e^{iL_{\mathcal{R}}s}$ and therefore $U(e^{ls}) = e^{i(L_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{R}} + \mathbb{1}_{\mathcal{S}} \otimes L_{\mathcal{R}})}$. As unitary operators have norm 1, we obtain an inequality for each $U_{\mathcal{S}}(e^{ls_0})$,

$$\|\epsilon(l : s_0)\| := \|\left(\Gamma_{\rho_{\mathcal{R}}} \circ \Phi \circ \iota\right)(U_{\mathcal{S}}(e^{ls_0})) - \Lambda(U_{\mathcal{S}}(e^{ls_0}))\| \leq \|\Phi_{\rho_{\mathcal{R}}} - \Lambda\|_{\mathrm{channel}}.$$

$F(\rho_0, \rho_1)$ represents the fidelity between two states ρ_0 and ρ_1 defined by $F(\rho_0, \rho_1) := \mathrm{tr}[\sqrt{\rho_0^{1/2} \rho_1 \rho_0^{1/2}}]$. This quantity is positive and equals 1 if and only if $\rho_0 = \rho_1$ holds.

Theorem 1. *Let $L_{\mathcal{S}}$ and $L_{\mathcal{R}}$ be generators of unitary representations of e^{ls} $(s \in \mathbb{R})$ on $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{R}}$ for $l \in \mathfrak{G}$. Define $U_{\mathcal{S}}(l : s) := e^{iL_{\mathcal{S}}s}$ and $U_{\mathcal{R}}(l : s) = e^{iL_{\mathcal{R}}s}$. Then, for any $s_0 \in \mathbb{R}$, $\epsilon(l : s_0) := \left(\Gamma_{\rho_{\mathcal{R}}} \circ \Phi \circ \iota\right)(U_{\mathcal{S}}(l : s_0)) - \Lambda(U_{\mathcal{S}}(l : s_0))$ is bounded for all $s \in \mathbb{R}$ by:*

$$\begin{aligned}
&\|[\Lambda(U_{\mathcal{S}}(l : s_0)), U_{\mathcal{S}}(l : s)]\| \leq 2\|U_{\mathcal{S}}(l : s) - \mathbb{1}\|\|\epsilon(l : s_0)\| \\
&+ \left(\frac{1}{F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l : s)\rho_R U_{\mathcal{R}}(l : s)^*)^2} - 1\right)^{1/2} \Big((\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l : s_0))^* \Lambda(U_{\mathcal{S}}(l : s_0))\| + 2\|\epsilon(l : s_0)\|)^{1/2} \\
&+ (\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l : s_0))\Lambda(U_{\mathcal{S}}(l : s_0))^*\| + 2\|\epsilon(l : s_0)\|)^{1/2} \Big).
\end{aligned}$$

Before proving Theorem 1, we present some immediate implications. We first observe that the left hand side of the above inequality vanishes for covariant Λ, since

$$\begin{aligned} \|[\Lambda(U_{\mathcal{S}}(l:s_0)), U_{\mathcal{S}}(l:s)]\| &= \|U_{\mathcal{S}}(l:s)^*[\Lambda(U_{\mathcal{S}}(l:s_0)), U_{\mathcal{S}}(l:s)]\| \\ &= \|U_{\mathcal{S}}(l:s)^*\Lambda(U_{\mathcal{S}}(l:s_0))U_{\mathcal{S}}(l:s) - \Lambda(U_{\mathcal{S}}(l:s_0))\|, \end{aligned} \tag{4}$$

and

$$U_{\mathcal{S}}(l:s)^*\Lambda(U_{\mathcal{S}}(l:s_0))U_{\mathcal{S}}(l:s) = \Lambda(U_{\mathcal{S}}(l:s)^*U_{\mathcal{S}}(l:s_0)U_{\mathcal{S}}(l:s)) = \Lambda(U_{\mathcal{S}}(l:s_0)).$$

Therefore, there is no bound for approximating covariant channels $\mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$ by restrictions of covariant channels $\mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$. Indeed, any covariant Λ can trivially be written as the restriction of a covariant channel Φ on $\mathbf{B}(\mathcal{H})$, i.e., as $\Phi_{\rho_{\mathcal{R}}}$ for all $\rho_{\mathcal{R}}$, by setting $\Phi = \Lambda \otimes \mathrm{id}$. If Λ is a unitary channel, Theorem 1 takes a much simpler form.

Corollary 1. *Under the same assumptions as Theorem 1, but for unitary Λ, it holds that*

$$\begin{aligned} \|[\Lambda(U_{\mathcal{S}}(l:s_0)), U_{\mathcal{S}}(l:s)]\| \leq & 2\|U_{\mathcal{S}}(l:s) - \mathbb{1}\|\|\epsilon(l:s_0)\| \\ & + 2\left(\frac{1}{F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_R U_{\mathcal{R}}(l:s)^*)^2} - 1\right)^{1/2} \|\epsilon(l:s_0)\|^{1/2}. \end{aligned}$$

The proof follows from the observation that if Λ is a unitary channel (or indeed, multiplicative), then for any unitary operator $U \in \mathbf{B}(\mathcal{H}_{\mathcal{S}})$, we have $\Lambda(U)^*\Lambda(U) = \mathbb{1}$.

Therefore, we see that in order to make possible good agreement between Λ and $\Phi_{\rho_{\mathcal{R}}}$, a highly "asymmetric" reference state $\rho_{\mathcal{R}}$ is necessary, since $F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_{\mathcal{R}}U_{\mathcal{R}}(l:s)^*)$ must decrease rapidly with respect to $|s|$ as otherwise the left-hand side of the inequality can be large for non-covariant Λ.

Furthermore, this asymmetry, or *coherence* factor, can be bounded by the "spread" of the (symmetry) generator $L_{\mathcal{R}}$:

Corollary 2. *In the same scenario as Theorem 1, it holds that*

$$\begin{aligned} \|[\Lambda(U_{\mathcal{S}}(l:s_0)), L_{\mathcal{S}}]\| \leq & 2\|L_{\mathcal{S}}\|\|\epsilon(l:s_0)\| \\ & + (\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}})\Big((\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l:s_0))^*\Lambda(U_{\mathcal{S}}(l:s_0))\| + 2\|\epsilon(l:s_0)\|)^{1/2} \\ & + (\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l:s_0))\Lambda(U_{\mathcal{S}}(l:s_0))^*\| + 2\|\epsilon(l:s_0)\|)^{1/2}\Big), \end{aligned}$$

where $\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}} := \sqrt{\mathrm{tr}[\rho_{\mathcal{R}} L_{\mathcal{R}}^2] - \mathrm{tr}[\rho_{\mathcal{R}} L_{\mathcal{R}}]^2}$ represents the standard deviation of $L_{\mathcal{R}}$ in the state $\rho_{\mathcal{R}}$.

Corollary 3. *Under the same assumptions as Theorem 1, but for unitary Λ, it holds that*

$$\|[\Lambda(U_{\mathcal{S}}(l:s_0)), L_{\mathcal{S}}]\| \leq 2\|L_{\mathcal{S}}\|\|\epsilon(l:s_0)\| + 2\sqrt{2}(\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}})\|\epsilon(l:s_0)\|^{1/2}.$$

This immediately follows from Corollary 2. The inequality is easy to interpret. For non-covariant Λ which yields non-vanishing left-hand side, $\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}}$ must be large to attain small $\|\epsilon(l:s_0)\|$. Thus it implies that the reference system $\mathcal{R}$ must be large (macroscopic). This result has some qualitative similarity to the bounds obtained in [14, 15], where large size/coherence/energy fluctuation of the reference is shown to be necessary for implementing unitary dynamics.

We now present proofs of Theorem 1 and Corollary 2. To prove Theorem 1, we need the following lemma [11, 18]:

Lemma 1. *Consider a channel $\Gamma : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. If $A, B \in \mathbf{B}(\mathcal{H})$ satisfy $[A, B] = 0$, then*

$$\begin{aligned}\|[\Gamma(A), \Gamma(B)]\| \leq & \|\Gamma(A^*A) - \Gamma(A)^*\Gamma(A)\|^{1/2}\|\Gamma(BB^*) - \Gamma(B)\Gamma(B)^*\|^{1/2} \quad (5)\\ & +\|\Gamma(AA^*) - \Gamma(A)\Gamma(A)^*\|^{1/2}\|\Gamma(B^*B) - \Gamma(B)^*\Gamma(B)\|^{1/2}. \quad (6)\end{aligned}$$

We now present the proof of Theorem 1.

Proof. If the state $\rho_\mathcal{R}$ and $s \in \mathbb{R}$ satisfy $F(\rho_\mathcal{R}, U_\mathcal{R}(l : s)\rho_\mathcal{R}U_\mathcal{R}(l : s)^*) = 0$, the claim follows trivially, and thus we assume otherwise. For notational simplicity, we omit the dependence on l and write $U_\mathcal{S}(s)$ for $U_\mathcal{S}(l : s)$, $U_\mathcal{R}(s)$ for $U_\mathcal{R}(l : s)$ and $\epsilon(s_0)$ for $\epsilon(l : s_0)$. We first write

$$[\Lambda(U_\mathcal{S}(s_0)), U_\mathcal{S}(s)] = [U_\mathcal{S}(s), \epsilon(s_0)] + [\Gamma_{\rho_\mathcal{R}}\Phi(U_\mathcal{S}(s_0) \otimes \mathbb{1}_\mathcal{R}), U_\mathcal{S}(s)]. \quad (7)$$

The first term on the right hand side is bounded as

$$\|[U_\mathcal{S}(s), \epsilon(s_0)]\| = \|[U_\mathcal{S}(s) - \mathbb{1}, \epsilon(s_0)]\| \leq 2\|U_\mathcal{S}(s) - \mathbb{1}\|\|\epsilon(s_0)\|.$$

To estimate the second term on the right hand side of (7), we introduce a purification of $\rho_\mathcal{R}$ to $|\phi_{RZ}\rangle \in \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_Z$, where we choose the purification space $\mathcal{H}_Z$ to be minimal, i.e., its dimension coincides with the rank of $\rho_\mathcal{R}$. We denote $\Gamma_{|\phi_{RZ}\rangle\langle\phi_{RZ}|} : \mathbf{B}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_Z) \to \mathbf{B}(\mathcal{H}_\mathcal{S})$ by Γ for simplicity. Now, for an arbitrary operator W_Z on $\mathcal{H}_Z$, we have

$$\Gamma(U_\mathcal{S}(s) \otimes U_\mathcal{R}(s) \otimes W_Z) = U_\mathcal{S}(s)\langle\phi_{RZ}|U_\mathcal{R}(s) \otimes W_Z|\phi_{RZ}\rangle.$$

In the following we denote $\Phi \otimes \mathrm{id}_Z$ by $\hat{\Phi}$, and in order to simplify some long expressions we will make the abbreviations $A_0 = U_\mathcal{S}(s_0) \otimes \mathbb{1}_\mathcal{R} \otimes \mathbb{1}_Z$ and $A_s = U_\mathcal{S}(s) \otimes U_\mathcal{R}(s) \otimes W_Z$ when convenient. Thus we have, for W_Z with $\langle\phi_{RZ}|U_\mathcal{R}(s) \otimes W_Z|\phi_{RZ}\rangle \neq 0$,

$$[\Gamma(\hat{\Phi}(A_0)), U_\mathcal{S}(s)] = \frac{[\Gamma(\hat{\Phi}(A_0)), \Gamma(A_s)]}{\langle\phi_{RZ}|U_\mathcal{R}(s) \otimes W_Z|\phi_{RZ}\rangle}.$$

Since Φ is a covariant channel, it holds that

$$(U_\mathcal{S}(s)^* \otimes U_\mathcal{R}(s)^*)\Phi(U_\mathcal{S}(s_0) \otimes \mathbb{1}_\mathcal{R})(U_\mathcal{S}(s) \otimes U_\mathcal{R}(s)) = \Phi(U_\mathcal{S}(s_0) \otimes \mathbb{1}_\mathcal{R}).$$

Therefore we find

$$[\Phi(U_\mathcal{S}(s_0) \otimes \mathbb{1}_\mathcal{R}) \otimes \mathbb{1}_Z, U_\mathcal{S}(s) \otimes U_\mathcal{R}(s) \otimes W_Z] = 0,$$

which enables us to apply Lemma 1. In the following, W_Z is chosen to be unitary. Now we bound

$$\|[\Gamma(\hat{\Phi}(A_0)), U_\mathcal{S}(s)]\| = \frac{\|[\Gamma(\hat{\Phi}(A_0)), \Gamma(A_s)]\|}{|\langle\phi_{RZ}|U_\mathcal{R}(s) \otimes W_Z|\phi_{RZ}\rangle|}.$$

Then Lemma 1 yields the numerator of the above equation to be bounded as

$$\begin{aligned}\|[\Gamma(\hat{\Phi}(A_0)), \Gamma(A_s)]\| \leq & \|\Gamma(\hat{\Phi}(A_0)^*\hat{\Phi}(A_0)) - \Gamma(\hat{\Phi}(A_0))^*\Gamma(\hat{\Phi}(A_0))\|^{1/2}\|\mathbb{1} - \Gamma(A_s)\Gamma(A_s)^*\|^{1/2}\\ & +\|\Gamma(\hat{\Phi}(A_0)\hat{\Phi}(A_0)^*) - \Gamma(\hat{\Phi}(A_0))\Gamma(\hat{\Phi}(A_0))^*\|^{1/2}\|\mathbb{1} - \Gamma(A_s)^*\Gamma(A_s)\|^{1/2}.\end{aligned}$$

We first estimate the norm of

$$A := \Gamma(\hat{\Phi}(A_0)^*\hat{\Phi}(A_0)) - \Gamma(\hat{\Phi}(A_0)^*\Gamma(\hat{\Phi}(A_0)).$$

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Due to the two-positivity of Γ (i.e., $\Gamma(X^*X) \geq \Gamma(X)^*\Gamma(X)$ for all X) the operator A is positive. Furthermore applying the two-positivity of $\hat{\Phi}$, we obtain

$$\hat{\Phi}(A_0)^*\hat{\Phi}(A_0) \leq \hat{\Phi}(A_0^*A_0) = \mathbb{1}.$$

Since Γ is a positive map we find

$$\mathbf{0} \leq A \leq \mathbb{1} - \Gamma(\hat{\Phi}(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}} \otimes \mathbb{1}_Z))^*\Gamma(\hat{\Phi}(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}} \otimes \mathbb{1}_Z)),$$

from which we conclude

$$\|A\| \leq \|\mathbb{1} - \Gamma(\hat{\Phi}(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}} \otimes \mathbb{1}_Z))^*\Gamma(\hat{\Phi}(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}} \otimes \mathbb{1}_Z))\|.$$

The term

$$\|\Gamma(\hat{\Phi}(A_0)\hat{\Phi}(A_0)^*) - \Gamma(\hat{\Phi}(A_0))\Gamma(\hat{\Phi}(A_0))^*\|^{1/2}$$

can be treated similarly. Writing $c_{RZ} \equiv \langle\phi_{RZ}|U_{\mathcal{R}}(s) \otimes W_Z|\phi_{RZ}\rangle$, we thus obtain

$$\|[\Gamma(\hat{\Phi}(A_0)), U_{\mathcal{S}}(s)]\| \leq \frac{1}{|c_{RZ}|}\Big(\|\mathbb{1} - \Gamma(\hat{\Phi}(A_0))^*\Gamma(\hat{\Phi}(A_0))\|^{1/2}\|\mathbb{1} - \Gamma(A_s)\Gamma(A_s)^*\|^{1/2} + \|\mathbb{1} - \Gamma(\hat{\Phi}(A_0))\Gamma(\hat{\Phi}(A_0))^*\|^{1/2}\|\mathbb{1} - \Gamma(A_s)^*\Gamma(A_s)\|^{1/2}\Big),$$

which is bounded above by

$$\frac{1}{|c_{RZ}|}(1 - |c_{RZ}|^2)^{1/2}\Big(\|\mathbb{1} - \Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))^*\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))\|^{1/2} + \|\mathbb{1} - \Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))^*\|^{1/2}\Big).$$

We estimate

$$\begin{aligned}\|\mathbb{1}-&\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))^*\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))\| \\ &= \|\mathbb{1} - \Lambda(U_{\mathcal{S}}(s_0))^*\Lambda(U_{\mathcal{S}}(s_0)) - \epsilon(s_0)^*\epsilon(s_0) - \epsilon(s_0)^*\Lambda(U_{\mathcal{S}}(s_0)) - \Lambda(U_{\mathcal{S}}(s_0))^*\epsilon(s_0)\| \\ &\leq \|\mathbb{1} - \Lambda(U_{\mathcal{S}}(s_0))^*\Lambda(U_{\mathcal{S}}(s_0))\| + 2\|\epsilon(s_0)\|.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\|\mathbb{1}-&\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))\Gamma_{\rho_{\mathcal{R}}}(\Phi(U_{\mathcal{S}}(s_0) \otimes \mathbb{1}_{\mathcal{R}}))^*\| \\ &\leq \|\mathbb{1} - \Lambda(U_{\mathcal{S}}(s_0))\Lambda(U_{\mathcal{S}}(s_0))^*\| + 2\|\epsilon(s_0)\|.\end{aligned}$$

Finally, one can choose W_Z so as to maximize $|\langle\phi_{RZ}|U_{\mathcal{R}}(s) \otimes W_Z|\phi_{RZ}\rangle|$, which coincides with $F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(s)\rho_R U_{\mathcal{R}}(s)^*)$ due to Uhlmann's theorem [23], thereby completing the proof. □

We now provide a proof of Corollary 2.

Proof. Adopting the shorthand $F \equiv F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_R U_{\mathcal{R}}(l:s)^*)$, the equality (4) replaces the inequality of Theorem 1 by,

$$\begin{aligned}\|U_{\mathcal{S}}(l:s)^*\Lambda(U_{\mathcal{S}}(l:s_0))U_{\mathcal{S}}(l:s) - \Lambda(U_{\mathcal{S}}(l:s_0))\| &\leq 2\|U_{\mathcal{S}}(l:s) - \mathbb{1}\|\|\epsilon(l:s_0)\| \\ &+ \Big(\frac{1}{F^2} - 1\Big)^{1/2}\Big((\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l:s_0))^*\Lambda(U_{\mathcal{S}}(l:s_0))\| + 2\|\epsilon(l:s_0)\|)^{1/2} \\ &+ (\|\mathbb{1}_{\mathcal{S}} - \Lambda(U_{\mathcal{S}}(l:s_0))\Lambda(U_{\mathcal{S}}(l:s_0))^*\| + 2\|\epsilon(l:s_0)\|)^{1/2}\Big).\end{aligned}$$

To bound the first term on the right hand side we write

$$U_{\mathcal{S}}(s) = \mathbb{1}_{\mathcal{S}} + i\int_0^s dt U_{\mathcal{S}}(t)L_{\mathcal{S}},$$

and therefore

$$\|U_{\mathcal{S}}(s) - \mathbb{1}_{\mathcal{S}}\| \leq |s|\|L_{\mathcal{S}}\|.$$

For the second term, we bound $F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_{\mathcal{R}}U_{\mathcal{R}}(l:s)^*)$ by choosing a purification of $\rho_{\mathcal{R}}$ as $|\phi\rangle \in \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_Z$. Then Uhlmann's theorem states that the fidelity is written as

$$F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_{\mathcal{R}}U_{\mathcal{R}}(l:s)^*) = \sup_{|\phi\rangle} |\langle\phi|e^{iL_{\mathcal{R}}s} \otimes \mathbb{1}_Z|\phi\rangle|.$$

For each purification $|\phi\rangle$, the Mandelstam-Tamm uncertainty relation [19, 20] provides a bound for $0 \leq \Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}} \cdot s \leq \pi/2$,

$$|\langle\phi|e^{iL_{\mathcal{R}}s} \otimes \mathbb{1}_Z|\phi\rangle| \geq \cos(\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}} \cdot s).$$

Thus we obtain

$$\left(\frac{1}{F(\rho_{\mathcal{R}}, U_{\mathcal{R}}(l:s)\rho_R U_{\mathcal{R}}(l:s)^*)^2} - 1\right)^{1/2} \leq \tan(\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}} \cdot s).$$

We divide the both terms by $|s|$ and take $|s| \to 0$ to obtain,

$$\begin{aligned}\|[\Lambda(U_{\mathcal{S}}(s_0)), L_{\mathcal{S}}]\| \leq 2\|L_{\mathcal{S}}\|\|\epsilon(s_0)\| + \Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}}\Big(&(\|\mathbb{1} - \Lambda(U_{\mathcal{S}}(s_0))^*\Lambda(U_{\mathcal{S}}(s_0))\| + 2\|\epsilon(s_0)\|)^{1/2} \\ &+ (\|\mathbb{1} - \Lambda(U_{\mathcal{S}}(s_0))\Lambda(U_{\mathcal{S}}(s_0))^*\| + 2\|\epsilon(s_0)\|)^{1/2}\Big).\end{aligned}$$

□

4. Rotational symmetry

As an example of the general behaviour we have investigated, we consider the possible dynamics of a qubit with Hilbert space $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$ under $SO(3)$ symmetry, realized by a true irreducible unitary representation of its universal covering group $SU(2)$. Since only a trivial unitary operator proportional to $\mathbb{1}$ commutes with all $SU(2)$ generators (angular momenta), one cannot change the state of the qubit in isolation (i.e., unitarily). The environment $\mathcal{H}_{\mathcal{R}}$ also has $SU(2)$ as a symmetry. We denote the angular momenta of the system and the reference frame by s_j and S_j ($j = x, y, z$) respectively. We consider an $SU(2)$-covariant channel $\Phi : \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$. The following corollary is immediately obtained from Corollary 2.

Corollary 4. *Let G be a Lie group. For a covariant channel $\Phi : \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}})$, its restriction $R \equiv \Gamma_{\rho_{\mathcal{R}}} \circ \Phi \circ \iota : \mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$, satisfies*

$$\|[R(U_{\mathcal{S}}(s_0)), L_{\mathcal{S}}]\| \leq (\Delta_{\rho_{\mathcal{R}}} L_{\mathcal{R}})\Big(\|\mathbb{1} - R(U_{\mathcal{S}}(s_0))^* R(U_{\mathcal{S}}(s_0))\|^{1/2} + \|\mathbb{1} - R(U_{\mathcal{S}}(s_0))R(U_{\mathcal{S}}(s_0))^*\|^{1/2}\Big).$$

We apply this to the case $G = SU(2)$ and $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$. For $l = s_x$, $U_{\mathcal{S}}(s)$ is written as $U_{\mathcal{S}}(s) = e^{is_x s} = e^{i\frac{\sigma_x}{2}s}$. We set $s_0 = \pi$ to obtain $U_{\mathcal{S}}(s_0) = i\sigma_x$. Then the restriction to the system $\Lambda := \Gamma_{\rho_{\mathcal{R}}} \circ \Phi \circ \iota : \mathbf{B}(\mathcal{H}_{\mathcal{S}}) \to \mathbf{B}(\mathcal{H}_{\mathcal{S}})$ satisfies the following three inequalities:

$$\begin{aligned}\|[\Lambda(\sigma_x), s_x]\| &\leq 2(\Delta_{\rho_{\mathcal{R}}} S_x)\|\mathbb{1} - \Lambda(\sigma_x)^2\|^{1/2} \\ \|[\Lambda(\sigma_y), s_y]\| &\leq 2(\Delta_{\rho_{\mathcal{R}}} S_y)\|\mathbb{1} - \Lambda(\sigma_y)^2\|^{1/2} \\ \|[\Lambda(\sigma_z), s_z]\| &\leq 2(\Delta_{\rho_{\mathcal{R}}} S_z)\|\mathbb{1} - \Lambda(\sigma_z)^2\|^{1/2},\end{aligned}$$

where $s_x = \frac{1}{2}\sigma_x$, *etc.* The uncertainty relations for angular momenta gives a non-trivial bound on sums of their fluctuations. We consider

$$\begin{aligned}(\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 &= \langle S_x^2 + S_y^2 + S_z^2 \rangle - (\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2) \\ &\leq l(l+1) - (\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2),\end{aligned}$$

where l is the magnitude of the largest spin of the environment. (Note that $\mathcal{H}_\mathcal{R}$ is written as a direct sum of irreducible representations of $SU(2)$ as $\mathcal{H}_\mathcal{R} = \oplus_s \mathbb{C}^{2s+1}$. l is the largest value of s in the summation.) It is easy to show that $\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2$ is rotationally invariant. We consider the quantity $\langle \mathbf{S} \cdot \mathbf{n} \rangle$ for $|\mathbf{n}| = 1$. This is a smooth function over the sphere and therefore has a maximum value at a certain point. To estimate the value of $\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2$, we assume that the maximum of $\langle \mathbf{S} \cdot \mathbf{n} \rangle$ is attained at $\mathbf{n} = \mathbf{e}_z$. By differentiating in polar coordinates, one can conclude that this state shows $\langle S_x \rangle = \langle S_y \rangle = 0$. Thus we have $\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 = \langle S_z \rangle^2$. Using $0 \leq \langle S_z \rangle^2 \leq l^2$, we conclude that

$$l \leq (\Delta S_x)^2 + (\Delta S_y)^2 + (\Delta S_z)^2 \leq l(l+1).$$

Thus we obtain the bound

$$\begin{aligned}&\|[\Lambda(\sigma_x), \sigma_x]\| + \|[\Lambda(\sigma_y), \sigma_y]\| + \|[\Lambda(\sigma_z), \sigma_z]\| \\ &\leq 2\sqrt{l(l+1)}(\|\mathbb{1} - \Lambda(\sigma_x)^2\| + \|\mathbb{1} - \Lambda(\sigma_y)^2\| + \|\mathbb{1} - \Lambda(\sigma_z)^2\|)^{1/2},\end{aligned}$$

where we used the Cauchy-Schwarz inequality. One can confirm, as expected, that any realizable non-covariant channel is inevitably dissipative, as non-dissipative (=unitary) dynamics satisfies $\mathbb{1} = \Lambda(\sigma_x)^2 = \Lambda(\sigma_y)^2 = \Lambda(\sigma_z)^2$. The right-hand side can be regarded as a quantity measuring the "dissipativity" of Λ, while the left-hand side represents the "magnitude" of dynamics. If the environment consists of N qubits, as $l = \frac{N}{2}$ holds the term $\sqrt{l(l+1)}$ in the right-hand side of the above inequality is proportional to N. Thus for Λ whose magnitude of dynamics is $O(1)$, its dissipativity cannot be smaller than $O\left(\frac{1}{N}\right)$ in the presence of N environment qubits.

We employ Stokes parameterization [24] to illustrate possible channels. Any qubit state is written as $\rho = \frac{1}{2}(\mathbb{1}_\mathcal{S} + \mathbf{x} \cdot \sigma)$ with $|\mathbf{x}| \leq 1$. Λ^*, the dual of Λ, maps ρ to another state $\rho' = \frac{1}{2}(\mathbb{1}_\mathcal{S} + \mathbf{y} \cdot \sigma)$. This map $(1, \mathbf{x}) \mapsto (1, \mathbf{y})$ is a linear map on $\mathbf{R}^4$ since Λ is self-adjoint. We denote this map by ${}^t\tilde{T}_\Lambda$ with a parameterization,

$${}^t\tilde{T}_\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & t_{11} & t_{12} & t_{13} \\ t_2 & t_{21} & t_{22} & t_{23} \\ t_3 & t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & T \end{pmatrix},$$

where T is a 3×3 matrix. Back in the Heisenberg picture, we obtain

$$\Lambda(a_0 \mathbb{1}_\mathcal{S} + \mathbf{a} \cdot \sigma) = (a_0 + \mathbf{t} \cdot \mathbf{a})\mathbb{1}_\mathcal{S} + (T\mathbf{a}) \cdot \sigma.$$

T can be written as

$$T = R_1 D R_2,$$

where R_1 and R_2 are elements of $SO(3)$ and D is a diagonal matrix as,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

One can choose the coordinate system so that $R_2 = \mathbb{1}$ is satisfied. Thus we will consider T with form $T = RD$. Then we obtain, for redefined $\mathbf{t}$,

$$\Lambda(a_0 \mathbb{1}_{\mathcal{S}} + \mathbf{a} \cdot \sigma) = (a_0 + \mathbf{t} \cdot \mathbf{a}) \mathbb{1}_{\mathcal{S}} + \sum_{ij=1}^{3} R_{ij} \lambda_j a_j \sigma_i,$$

where $R_{ij} \in SO(3)$. Assume that R is written as a rotation around the z-axis, with the vector $\mathbf{t} = \mathbf{0}$, as,

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\lambda_x |\sin\theta| \leq (\Delta_{\rho_{\mathcal{R}}} S_x) \sqrt{1 - \lambda_x^2}$$
$$\lambda_y |\sin\theta| \leq (\Delta_{\rho_{\mathcal{R}}} S_y) \sqrt{1 - \lambda_y^2}.$$

That is, we have a relation between the dissipative and symmetry breaking natures.

$$\lambda_x^2 \leq \frac{(\Delta_{\rho_{\mathcal{R}}} S_x)^2}{(\Delta_{\rho_{\mathcal{R}}} S_x)^2 + (\sin\theta)^2};$$
$$\lambda_y^2 \leq \frac{(\Delta_{\rho_{\mathcal{R}}} S_y)^2}{(\Delta_{\rho_{\mathcal{R}}} S_y)^2 + (\sin\theta)^2}.$$

5. Concluding remarks

We have seen that there is a positive lower bound on the difference between an arbitrary quantum channel and the restriction of a covariant channel, and moreover, that in order to reduce this discrepancy a large spread in the generator of the symmetry is needed in the reference system. This result bears similarities with the WAY theorem, and is in line with the relational view of quantum mechanics, wherein we interpret non-symmetric channels as representatives of their symmetric counterparts of system and reference taken together. The large spread required for good approximation of relative (symmetric) by non-relative (asymmetric) can be understood as a condition on the quality of the reference frame, in the sense of the findings of [3] and [11]. As a final remark, we mention that there is yet another symmetry condition on channels that differs from the one employed in this paper and arises naturally in the context of quantum reference frames. We will return to this issue elsewhere.

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