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Mahmoud A. Farghaly, Ulrik Hanke, Muhammad N. Akram, Einar Halvorsen
Department of Microsystems
University of Southeast Norway
Raveien 215
3184 Borre, Norway

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Trial functions for reduced-order models of piezoelectrically actuated MEMS tunable lenses

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*Department of Microsystems, University of Southeast Norway, Raveien 215, 3184, Borre, Norway

Abstract. Piezoelectrically actuated MEMS lens structures can be composed of a clamped square elastic diaphragm partially covered with a thin piezoelectric film leaving a circular transparent region to form a lens pupil. To model these lenses’ linear static optoelectromechanical performance, the displacement can be approximated by a linear combination of basis functions, e.g., weighted Gegenbauer polynomials that satisfy clamped boundary conditions along the diaphragm edges. However, such a model needs as much as 120 degrees of freedom (DOFs) to provide a good approximation of the lens optical performance. To improve on this, we here consider approximating the deflection by an expansion using piecewise smooth functions that have different forms in the pupil and the actuator regions. We use exact solutions for the elastic plate differential equation over circular and annular subdomains, and weighted Gegenbauer polynomials in the remaining region. The latter enforces the boundary conditions. We have found that the larger the diaphragm area with exact plate solutions is, the lower is the number of DOFs needed to predict mechanical and optical quantities accurately. For example, a model with 10 DOFs achieves accuracies of 5.1% and 2.1% respectively for RMS wavefront error and reciprocal F-number for all pupil openings of interest.

Keywords: Microelectromechanical systems, Adaptive optics, Lenses, Focus, Piezoelectric effect, Actuators, Imaging systems.

*Einar Halvorsen, einar.Halvorsen@usn.no

1 Introduction

Tunable focus in cameras enables capture of sharp images over a wide range of camera-object distances and is considered an essential feature in modern cameras. Voice coil motors and ultrasonic motors are widely commercialized macro-scale technologies for tuning focus in devices such as mobile cameras. However, microelectromechanical systems (MEMS) lenses promise low-power, small footprint mechanisms utilized for the same purpose.

In this paper, we consider piezoelectrically actuated MEMS tunable lenses. Such a lens has a diaphragm consisting of a square glass plate covered by a thin piezoelectric film with a circular opening in the film leaving the central part transparent. When the piezoelectric film is biased, the diaphragm bends and a plano-convex lens is formed by a high-refractive index polymer sandwiched between the diaphragm and a second transparent plate.
Mathematical models are necessary for design and optimization of such MEMS devices and involve solving coupled multiphysics problems. A possible solution to this challenge is to use numerical calculations based on the finite element methods (FEM). This approach results in a large number of degrees of freedom (DOFs) due to the discretization of the device geometry in finite element models, which has a significant drawback of long computational time. This is particularly pressing for modeling the transient behavior, but is also important when a large number of static cases are needed for optimization. For system-level designers to have computationally efficient models, it is necessary to develop reduced-order models that can be implemented by e.g. using MATLAB or a circuit simulator yet faithfully representing the device physics.

System-level models can be made few DOFs through model-order-reduction (MOR) techniques. These techniques aim for an efficient projection to eliminate several DOFs and maintain the ones sufficient to capture the system behavior. However, such an approach hinges on first solving a model with many DOFs, then reducing. The MOR approach solves the problem of obtaining a low order system-level model, but still requires a significant computational effort to obtain it. Therefore, it is of interest to have methods that, unlike the MOR techniques, give a low order model directly without relying on projection.

Low order models can in principle be obtained by analytical or semi-analytical (series expansion) solutions. For similar mechanical problems of a rectangular plate with a circular hole, which is a common substructure in naval and aircraft architectures, this can be done. However, these solutions are not applicable to the lens because the problems are different. In particular, the lens has additional physics due to the piezoelectric film and does not have a hole, but a circular purely elastic region instead.

To evaluate the linear static performance of piezoelectrically actuated lenses, we previously es-
established a modeling framework\textsuperscript{12} composed of two parts. The first part is a variational formulation that models the lens’ deformation due to piezoelectric actuation while the second part evaluates the lens’ optical parameters such as the $F$-number ($F\#$) and RMS-wavefront-error (RMSWFE). We represented the diaphragm deflection by an expansion in a weighted Gegenbauer basis\textsuperscript{12} using a formulation similar to,\textsuperscript{13} but including piezoelectricity and the elasticity of the circular part. In this case, each basis function is extended continuously over the entire diaphragm and 120 DOFs were necessary to reach a satisfactory representation of the lens optical performance. While this is a major improvement in computational effort compared to FEM, it is still a quite large number of DOFs for lumped-model system simulations and too large to be tractable by purely analytical means.

One weakness in the previous formulation was that the basis functions did not account for the discontinuity of the layered structure at the lens opening. There are good reasons to expect that an improvement in convergence could be achieved by taking this discontinuity into account. For example, the deflection of purely elastic circular plates due to asymmetric bending\textsuperscript{14,15} was previously modeled using circular and annular FEM elements with analytical solutions of the biharmonic equation as interpolation functions instead of general ones such as hexahedral or tetrahedral elements. It improved convergence, reduced the requirements on mesh refinement and represented the curved boundary well.

Motivated by the previous solutions to the purely mechanical problem,\textsuperscript{11,13–15} this paper presents an approach that significantly improves model accuracy for the piezoelectrically actuated lens by using basis functions that account for the discontinuity in the layered structure at the lens opening, uses the exact solution of the biharmonic equation in the circular regions and fulfills the boundary conditions at the diaphragm edges.
We have chosen analytical ansätze that have Gegenbauer-polynomial-based subfunctions with rectangular symmetry satisfying the plate’s boundary conditions and yet can be expanded on the form of Fourier series along the circular discontinuity to be matched term-by-term with the exact solutions of the plate’s differential equation. Thus, the presented models can be generally used for any similar structures after reformulating the variational formulation to include the actuating forces due to e.g. pressure, piezoelectricity or thermoelasticity. For our lens application, the approach succeeds in reducing the model down to 10 DOFs as opposed to 120 for the same accuracy in the previous approach.

2 Principle of Operation

The piezoelectrically actuated MEMS lens is composed of a refractive polymer sandwiched between two transparent glass layers and with a piezoelectric stack on top (see Fig. 1). The upper glass layer is bent upwards due to the piezoelectric coupling whenever a DC voltage $V_p$ is applied across the piezoelectric stack. As a result, the soft polymer follows the plate deformation and forms a plano-convex lens focusing light rays. In this manner, the lens’s focal length can be tuned using the voltage $V_p$ in order to focus on objects at varying distances. This type of tunable lenses can be combined with a fixed-focal-length lens to tune the overall system focal length.

3 Normalized coordinates

The diaphragm of the considered lens is square with a side length $a$ and is clamped along its four sides. Figure 2 shows planar views of the lens marked with definitions used by different models. Models 0 and 1 break the lens domain $\Omega$ into two subdomains $\Omega_1$ and $\Omega_2$ while model 2 breaks it into 3 subdomains $\Omega_1$, $\Omega_{II}$ and $\Omega_{III}$ (we have assigned new labels for subdomains in model 2 to
simplify the mathematical representation of variables later on). The lens diaphragm extends over a square with cartesian coordinates $x, y \in [-a/2, a/2]$ and it is convenient to introduce normalized coordinates $X = 2x/a$ and $Y = 2y/a$. Thus, the locus of the lens pupil boundary ($\Gamma_{\Omega_1}$ in Fig. 2a or $\Gamma_{\Omega_1}$ in Fig. 2b) and the fictitious boundary $\Gamma_{\Omega_{II}}$ in these normalized cartesian coordinates are given by $\sqrt{X^2 + Y^2} = \gamma_1$ and $\sqrt{X^2 + Y^2} = \gamma_2$ where $\gamma_1$ and $\gamma_2$ respectively are the ratio of the lens pupil and the fictitious circle diameters to the diaphragm side length $a$.

The lens’ circular and annular subdomains $\Omega_1$, $\Omega_1$ and $\Omega_{II}$ can be further normalized to a radial coordinate, as shown in Fig. 3. For these subdomains, we use the normalized radial coordinate $r = \sqrt{X^2 + Y^2}/\gamma_0$ with $\gamma_0 = \gamma_1$ for models 0 and 1 and to $\gamma_0 = \gamma_2$ for model 2. As shown in Fig. 3, the lens pupil boundary for the different models is either of the circles $r = 1$ or $r = \alpha$ where $\alpha = \gamma_1/\gamma_2$.

4 Variational formulation and its integrals

We use the variational formulation formerly presented in Ref. 12. It is based on classical laminated plate theory, linear piezoelectricity, quasi-electrostatic conditions and thin film approximation. The lens thickness is much smaller than its lateral dimension which justifies applying the classical
Fig 2 Planar views of the piezoelectrically actuated MEMS tunable lens showing decomposing its structure into subdomains. (a) Models 0 and 1 break the lens into two subdomains: $\Omega_1$ and $\Omega_2$. (b) Model 2 breaks it into three subdomains: $\Omega_1$, $\Omega_{11}$ and $\Omega_{111}$. Subdomains $\Omega_{11}$ and $\Omega_{111}$ are separated by a fictitious circular boundary $\Gamma_{\Omega_{11}}$.

Fig 3 Planar views showing the normalized radial coordinates in the circular and annular subdomains for (a) models 0 and 1, and (b) model 2.
laminated plate theory. Thus, the model assumes that the deflection in z-direction \( w_0 \) is due to bending and shear strains are neglected. The middle-plane stretching and the corresponding axial in-plane displacements are also neglected.

For a piezoelectric medium, the principle of virtual work can be stated as

\[
\delta H - \delta W = 0
\]  

(1)

where \( \delta H \) is the virtual variation of the electrical enthalpy \( H = H(\{S_{ij}\}, \{E_k\}) \) and \( \delta W \) is the virtual variation of potential energy due to external applied forces. \( S_{ij} \) and \( E_k \) are components of strain and the electric field. The displacements in strain expressions are usually approximated by ansatz (discussed thoroughly in section 5) that is a linear combination of basis functions whose weights are to be determined. Equation (1), after substituting a displacement ansatz, is can be expressed as

\[
\sum_q R_{\Omega_q} - \sum_q F_{\Omega_q} = 0
\]  

(2)

where \( R_{\Omega_q} \) and \( F_{\Omega_q} \) are the spring and force variational energy integrals for the subdomain \( \Omega_q \). We will write these integrals on general forms to simplify the later presentation of the variational formulation of the three models in section 5. These general forms have arguments representing geometry, material parameters and displacement ansatz for each subdomain. Thus, the spring variational integrals over square (subscript □) and annular (or circular, subscript ◦) subdomains can be written as
\[ R_\square(w_0, \delta w_0; D^*) = \frac{1}{(a/2)^2} \int_{-1}^{1} \int_{-1}^{1} U^T D^* \delta U dX dY, \quad (3) \]

\[ R_o(w_0, \delta w_0; D^*, \gamma_0, \alpha_H, \alpha_L) = \frac{1}{\gamma_0^2 (a/2)^2} \int_{\alpha_L}^{\alpha_H} \int_{0}^{2\pi} V^T D^* \delta V r dr d\theta \quad (4) \]

where \[
U = \begin{bmatrix}
w_{0,XX} \\
w_{0,YY} \\
2w_{0,XY}
\end{bmatrix}, \quad V = \begin{bmatrix}
w_{0,rr} \\
\left(\frac{1}{r} w_{0,r} + \frac{1}{r^2} w_{0,\theta}\right) \\
2\left(\frac{1}{r} w_{0,r\theta} - \frac{1}{r^2} w_{0,\theta}\right)
\end{bmatrix}. \quad (5)\]

\(\alpha_L, \alpha_H \in [0, 1]\) represent the lower and upper limits for the integral over the normalized coordinate \(r\) for an annular (or circular) subdomain. The modified membrane flexural rigidity matrix is defined as

\[ D^* = \begin{bmatrix}
D^*_{11} & D^*_{12} & 0 \\
D^*_{12} & D^*_{22} & 0 \\
0 & 0 & D^*_{66}
\end{bmatrix}. \quad (6)\]

The modified flexural rigidities for each subdomain varies due to the difference in layer struc-
tures. For the tunable lens under study we have

**Models 0 & 1:**

\[ D_{ij}^{e,\Omega_1} = D_{ij}^{gl}, \]

\[ D_{ij}^{e,\Omega_2} = D_{ij}^{gl} + D_{ij}^{p}, \]

**Model 2:**

\[ D_{ij}^{e,\Omega_1} = D_{ij}^{gl}, \]

\[ D_{ij}^{e,\Omega_II} = D_{ij}^{e,\Omega_III} = D_{ij}^{gl} + D_{ij}^{p}, \]

where \( D_{ij}^{gl} \) and \( D_{ij}^{p} \) are respectively the flexural rigidities for the glass layer only and the piezoelectric layer only.

The force variational energy integrals over square and annular subdomains can be written as

\[ F_{\Box}(\delta w_0) = V_p z_p \bar{e}_{31} \int_{-1}^{1} \int_{-1}^{1} \nabla^2 \delta w_0 dX dY \]

\[ = V_p z_p \bar{e}_{31} \int_{\Gamma_{\Omega}} \nabla \delta w_0 \cdot d\hat{n} = 0, \quad (7) \]

\[ F_\circ(\delta w_0; \alpha_H, \alpha_L) = V_p z_p \bar{e}_{31} \int_{\alpha_L}^{\alpha_H} \int_{0}^{2\pi} \nabla^2 \delta w_0 r dr d\theta \quad (8) \]

where \( z_p = (h_3 + h_2)/2 \) (see Fig. 1b) and \( \bar{e}_{31} \) is the effective longitudinal e-form piezoelectric coupling coefficient.\(^\text{12}\) The 2-D Laplace operator \( \nabla^2 \) can be expressed in normalized cartesian or polar coordinates according to the shape of the domain. \( d\hat{n} \) is the differential length vector normal to the lens outer boundary \( \Gamma_{\Omega} \). Using Green’s theorem, we have written the surface integral in Eq. (7) as a line integral over the closed boundary \( \Gamma_{\Omega} \). The displacement anstaz inside that integral must satisfy the lens clamping conditions of zero slope along the four membrane sides, which
mandates Eq. (7) to be always equal to 0.

5 Models

This section describes mathematically the basis functions in the lens subdomains and the linear system of equations that arises from the variational formulation for three models. In order to have a good representation in each subdomain, different functional forms can be used in each domain. For each subdomain, the displacement is specified as a subfunction that is a linear combination of basis functions that are specific to the domain. The subfunctions in neighboring subdomains are required to satisfy continuity of displacement, slope and, if needed, higher derivatives of displacement. Hence, the coefficients of the different subfunctions are not independent degrees of freedom. Instead, the coefficients of the inner domain(s) can be expressed uniquely in terms of the coefficients of the outer domain.

Ideally, the number of basis functions (or number of DOFs) should be infinite to span any displacement. However, the variational models use a finite number of these functions, which form a finite subspace of this infinitely-sized space. The dimension of this finite subspace is increased to improve accuracy of the predicted mechanical and optical quantities.

In the following analysis due to lens mirror symmetries, we will only consider 4-fold symmetric trial functions from Gegenbauer polynomials and the homogeneous solution of the plate differential equations to be displacement ansatz.

5.1 Model 0

This model from uses weighted Gegenbauer polynomials as basis functions over all the lens planar subdomains. These polynomials do not satisfy the plate differential equations, but they sat-
isfy the clamped boundary conditions of zero displacement and slope along the square diaphragm edges. Moreover, they are orthogonal polynomials and are easily mapped to Zernike polynomials which are suitable for optical wavefront representation. With these basis functions, 120 independent DOFs were needed to obtain a satisfying representation of the lens’s optical performance from this variational model when compared to FEM simulations.

5.1.1 Weighted Gegenbauer Polynomials

These basis functions are orthogonal functions in the normalized variables $X$ and $Y$. They are products of a weight factor $(X^2 - 1)^2(Y^2 - 1)^2$ and two Gegenbauer polynomials; one is a function of $X$ and the other a function of $Y$. The weight factor enforces zero displacement and slope along the square membrane edges. A function in the basis is conveniently denoted $\phi_{lj}$ where the first index denotes the $X$-dependent Gegenbauer polynomial and the second one denotes the $Y$-dependent polynomial. Due to the $90^\circ$ rotational symmetry of the lens structure, not all combinations of basis functions are possible. We can reduce the basis to a single-index set of functions $\Phi_k$ given by

$$
\Phi_k(X,Y) = \frac{1}{2} [\phi_{lj}(X,Y) + \phi_{jl}(X,Y)]
$$

(9)

where only even indices $l, j = 0, 2, \cdots N - 1$ occur; $N$ is an odd number representing the number of Gegenbauer polynomials for either of the variables $X$ or $Y$. The index $k = 1, 2, \cdots, N_G = \frac{1}{8}(N + 1)(N + 3)$ enumerates the single-index basis functions. The label $k$ is obtained from the indices $l, j$ by counting along the zigzag trajectory shown in Fig. 4. This simplification reduces the number of basis functions from $\frac{1}{4}(N + 1)^2$ to $N_G$ which is nearly a factor 2 for large $N$ values.
The first three $\Phi_k$ are

$$\Phi_1(X,Y) = \phi_{00} = (X^2 - 1)^2(Y^2 - 1)^2, \quad (10)$$

$$\Phi_2(X,Y) = \frac{1}{2}(\phi_{02} + \phi_{20})$$

$$= \frac{9}{4}\phi_{00}(11X^2 + 11Y^2 - 2), \quad (11)$$

$$\Phi_3(X,Y) = \phi_{22}$$

$$= \frac{81}{4}\phi_{00}(11X^2 - 1)(11Y^2 - 1). \quad (12)$$

For the lens’ subdomains with circular symmetry, it is sometimes more convenient to use polar coordinates $r$ and $\theta$ such that $(X,Y) = (\gamma_0 r \cos \theta, \gamma_0 r \sin \theta)$ and the basis functions are
\( \widetilde{\Phi}_k(r, \theta, \gamma_0) = \Phi_k(\gamma_0 r \cos \theta, \gamma_0 r \sin \theta) \). Using MATLAB symbolic toolbox, the first three \( \widetilde{\Phi}_k \) are

\[
\widetilde{\Phi}_1 = \frac{3}{128} \gamma_0^8 r^8 - \frac{1}{4} \gamma_0^6 r^6 + \frac{5}{4} \gamma_0^4 r^4 - 2 \gamma_0^2 r^2 + 1
- \frac{1}{32} \gamma_0^4 r^4 (\gamma_0^4 r^4 - 8 \gamma_0^2 r^2 + 8) \cos(4 \theta)
+ \frac{1}{128} \gamma_0^8 r^8 \cos(8 \theta),
\]

(13)

\[
\widetilde{\Phi}_2 = \frac{9}{512} (11 \gamma_0^2 r^2 - 2) \left[ (3 \gamma_0^8 r^8 - 32 \gamma_0^6 r^6 + 160 \gamma_0^4 r^4
- 256 \gamma_0^2 r^2 + 128)
- 2 \gamma_0^4 r^4 (\gamma_0^4 r^4 - 8 \gamma_0^2 r^2 + 8) \cos(4 \theta)
+ \gamma_0^8 r^8 \cos(8 \theta) \right],
\]

(14)

\[
\widetilde{\Phi}_3 = \frac{49005}{4096} \gamma_0^{12} r^{12} - \frac{61479}{512} \gamma_0^{10} r^{10} + \frac{244377}{512} \gamma_0^8 r^8
- \frac{14337}{16} \gamma_0^6 r^6 + \frac{24867}{32} \gamma_0^4 r^4 - \frac{1053}{4} \gamma_0^2 r^2 + \frac{81}{4}
- \frac{81}{8192} \gamma_0^4 r^4 \left( 1815 \gamma_0^8 r^8 - 16192 \gamma_0^6 r^6
+ 52160 \gamma_0^4 r^4 - 68096 \gamma_0^2 r^2 + 31488 \right) \cos(4 \theta)
+ \frac{81}{4096} \gamma_0^8 r^8 \left( 363 \gamma_0^4 r^4 - 2024 \gamma_0^2 r^2
+ 1944 \right) \cos(8 \theta) - \frac{9801}{8192} \gamma_0^{12} r^{12} \cos(12 \theta).
\]

(15)

From Eqs. (13-15), each basis function has the form of a Fourier cosine series that can be
expressed as

\[ \Phi_k(X, Y) = \tilde{\Phi}_k(r, \theta, \gamma_0) = f_{k,0}(r; \gamma_0) \]

\[ + \sum_{n=2,4,6,\ldots}^{N_{s,k}} f_{k,n}(r; \gamma_0) \cos(n\theta) \]  

(16)

where \( n \) is the Fourier series index. \( f_{k,n}(r; \gamma_0) \) is the \( n^{th} \) Fourier coefficient for the function \( \tilde{\Phi}_k \) and is dependent on the normalized radial coordinate \( r \) and the ratio \( \gamma_0 \). Due to the lens’ 4-fold symmetry, the coefficients \( f_{k,n}(r; \gamma_0) \) with \( n = 2, 6, 10, \ldots \) are zero. \( N_{s,k} \) is the number of Fourier terms used to expand the function \( \tilde{\Phi}_k \).

To represent any basis function \( \phi_{lj} \) by a Fourier series, the Fourier terms need to be up to \( N_{s,lj} = l + j + 8 \). Thus, the highest order in Fourier terms for a function set of \( N_G \) single-index Gegenbauer polynomials is \( N_F = \max(N_{s,lj}) = 2(N - 1) + 8 \). Table 1 shows examples of the corresponding number \( N_F \) and index \( k \) values for a certain number of Gegenbauer polynomials \( N \).

In that manner, a function set of \( N_G \) single-index basis functions form a finite Gegenbauer space \( \mathbb{R}^{N_G} \) that can be mapped to a Fourier space \( \mathbb{R}^{N_F/2+1} \) using Eq. (16).

**Table 1** Examples on Gegenbauer polynomials order \( N \) with its correspondent values for the single index \( k \) and the number of sufficient Fourier terms \( N_F \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 \rightarrow N_G )</td>
<td>1→3</td>
<td>1→6</td>
<td>1→10</td>
<td>1→15</td>
<td></td>
</tr>
<tr>
<td>( N_F = \max(N_{s,lj}) )</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
</tr>
</tbody>
</table>

Model 0 approximates the lens displacement with a finite linear combination of basis functions
Φ_k(X, Y)

\[ w_0(X, Y) = \tilde{w}_0(r, \theta, \gamma_0) = \sum_{k=1}^{N_G} C_k \Phi_k(X, Y) \]

\[ = \sum_{k=1}^{N_G} C_k \tilde{\Phi}_k(r, \theta, \gamma_0) \] (17)

where \( C_k \) are the coefficients to be determined.

5.1.2 Variational formulation

After substituting with the displacement ansatz from Eq. (17) in the variational formulation of Eq. (2), it becomes

\[ R_{\Omega_1} + R_{\Omega_2} = F_{\Omega_1} + F_{\Omega_2}, \] (18)

such that

\[ R_{\Omega_1} = R_c(w_0, \delta w_0; D_{\Omega_1}, \gamma_1, 1, 0) = \delta C^T R_{\Omega_1} C, \] (19)

\[ R_{\Omega_2} = R_{\square}(w_0, \delta w_0; D_{\Omega_2}^s) \\
- R_c(w_0, \delta w_0; D_{\Omega_1}, \gamma_1, 1, 0) = \delta C^T R_{\Omega_2} C, \] (20)

\[ F_{\Omega_1} = 0, \] (21)

\[ F_{\Omega_2} = F_{\square}(\delta w_0) - F_c(\delta w_0, 1, 0) = \delta C^T F_{k_2}, \] (22)

where \( C \) is a column vector of the DOFs \( C_k \). Expressions of the elements in the matrices \( R_{\Omega_q}^{\Omega_{k_1 k_2}} \) \((q = 1, 2)\) and \( F_{k_2} \) are listed in Appx. A. The planar subdomain \( \Omega_2 \) is prismatic and can be imagined to result from subtracting a domain with a circular base face from another one with a
square base face (see Fig. 2a). From that aspect, integrals over that subdomain are easily calculated as a difference between two integrals as in Eqs. (20) and (22); the first is over a square area and the second is over a circular area.

By combining the variational terms from Eqs. (19-22) and substitute in Eq. (18), model-0’s variational formulation turns to be

\[ \delta C^T (R_{k_1, k_2}^{\Omega_1} + R_{k_1, k_2}^{\Omega_2}) C = \delta C^T F_{k_2}. \]  

(23)

The above variational formulation is solved if the coefficient vector \( C \) is a non-trivial solution to the linear system of equations

\[ (R_{k_1, k_2}^{\Omega_1} + R_{k_1, k_2}^{\Omega_2}) C = F_{k_2}. \]  

(24)

Each term inside the parentheses on the left-hand side (LHS) represents the equivalent stiffness matrix of each lens subdomain. The right-hand side (RHS) represents the equivalent force due to the piezoelectric actuation.

5.2 Model 1

Model-1 deals with the lens as a 2-subdomains problem similar to model 0, but it uses a piecewise expansion of two different basis functions for the displacement approximation in the pupil and
actuator regions. Its displacement ansatz in the lens subdomains is

\[ \Omega_1: \quad w^{(1)}_0 = A^{1}_0 + B^{1}_0 r^2 \]
\[ + \sum_{n=2,4,6,\ldots}^{N_F} (A^{1}_n r^n + B^{1}_n r^{n+2}) \cos(n\theta), \quad (25) \]
\[ \Omega_2: \quad w^{(2)}_0 = \sum_{k=1}^{N_G} C_k \Phi_k(X,Y) \quad (26) \]

where \( w^{(1)}_0 \) is the subfunction of the displacement ansatz in the subdomain \( \Omega_1 \). It has \( N_L = N_F + 2 \) coefficients \( A^1_i \) and \( B^1_i \) for \( i = 0, 2, \ldots, N_F \) to be determined. Due to the lens symmetry, \( w^{(1)}_0 \) has only even terms of the general homogeneous solution for the plate differential equation.\(^{21}\) To have a bounded solution at the origin \( r = 0 \), we have eliminated logarithmic and negative-power terms from the homogeneous solution. Moreover, we have limited the upper value of the summation index in Eq. (25) to \( N_F \) in order to have the same trigonometric terms as in the weighted Gegenbauer basis from Eq. (26). This limitation follows from the continuity requirement on the displacement and the slope at the boundary \( \Gamma_{\Omega_1} \) separating the lens’ subdomains (thoroughly discussed in section 5.2.1). The ansatz part \( w^{(1)}_0 \) is equivalent to having a circular FEM element with interpolation functions formed as a product of two polynomials: one is an even polynomial in \( r \) for the radial direction and the other a cosine function for the circumferential direction.\(^{15}\)

For the subdomain \( \Omega_2 \), the subfunction \( w^{(2)}_0 \) is the same single-index Gegenbauer basis used in model 0 to enforce the clamped boundary conditions.

5.2.1 Relationships between coefficients at the pupil boundary \( \Gamma_{\Omega_1} \)

Both subfunctions of the ansatz in model 1 have the form of a Fourier series; sum of products of a radial factor and a trigonometric function. The radial factor multiplying each trigonometric
function of $w_0^{(1)}$ has two coefficients while those of $w_0^{(2)}$ have only one coefficient. To find a relation between these two sets of coefficients at $\Gamma_\Omega_1$, two boundary conditions are required: continuity of the displacement and of the slope. Hence, we equate the radial factor of trigonometric terms (term by term) of the pair $(w_0^{(1)}$, $w_0^{(1)}$, $r)$ to their counterpart radial factor of the trigonometric terms of $(w_0^{(2)}$, $w_0^{(2)}$, $r)$ at the boundary $\Gamma_\Omega_1$ where $\gamma_0 = \gamma_1$ and $r = 1$. In matrix form, the relation is compactly expressed as

$$L_{I|r=1}A_I = N_I \mathbf{C}$$

$$= \begin{bmatrix}
N_{1,I} & N_{2,I} & \cdots & N_{N_G,I}
\end{bmatrix} \begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_{N_G}
\end{bmatrix}.$$  \hspace{1cm} \text{(27)}$$

where

$$L_I = \begin{bmatrix}
L_{0,I} & 0 & 0 & \cdots & 0 \\
0 & L_{2,I} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & L_{N_F,I}
\end{bmatrix},$$ \hspace{1cm} \text{(28)}$$
\[
A_i = \begin{bmatrix}
  A_{0,i} \\
  A_{2,i} \\
  \vdots \\
  A_{N_F,i}
\end{bmatrix}, \quad A_{i,1} = \begin{bmatrix}
  A_i^1 \\
  B_i^1
\end{bmatrix}
\] (29)

\[
N_{k,1} = \begin{bmatrix}
  N_{k,0}^1 \\
  N_{k,2}^1 \\
  \vdots \\
  N_{k,N_{s,k}}^1 \\
  0
\end{bmatrix}
\] (30)

In Eq. (27), the column vector \(N_{k,1}\) of length \(N_L\) represents the radial factors and their derivatives of each basis function \(\Phi_k\). The index of Fourier terms for each function \(\Phi_k\) ranges from 0 to \(N_{s,k}\). Thus, the last \(N_L - N_{s,k} - 2\) elements of the \(N_{k,1}\) column are zeros while the first \(N_{s,k} + 2\) ones can be calculated from Eq. (16). The submatrices of \(L_{i,1}\) and \(N_{k,1}\) are given by

\[
L_{i,1} = \begin{bmatrix}
  r^i & r^{i+2} \\
  i r^{i-1} & (i + 2) r^{i+1}
\end{bmatrix},
\]

\[
N_{k,i}^{f'} = \begin{bmatrix}
  f_{k,i}(r = 1, \gamma_0 = \gamma_1) \\
  f_{k,i}'(r = 1, \gamma_0 = \gamma_1)
\end{bmatrix}
\] (31)

where the prime in \(f_{k,i}'\) denotes differentiation with respect to the variable \(r\). Rows of the product \(L_{i,1}A_{i,1}\) and \(\sum_{k=1}^{N_G} N_{k,i}^{f'} C_k\) respectively correspond to \(i^{th}\)-radial factor of the trigonometric terms of
the pairs \((w_0^{(1)}, w_0^{(1)})\) in Eq. (25) and \((w_0^{(2)}, w_0^{(2)})\) in Eqs. (16) and (26).

Since \(L_I\) is a square invertible matrix, Eq. (27) can be rewritten as

\[
A_I = L_I^{-1}|_{r=1} N_I C = T_1 C, \quad T_1 \in \mathbb{R}^{N_L \times N_G}
\]  

(32)

where \(T_1\) is model 1’s transformation matrix relating coefficients \(A_I\) and \(C\) in the adjacent subdomains. Appendix B lists an example of the matrices \(T_1\) and \(N_I\) using the first basis function \(\Phi_1\).

To numerically calculate \(L_I^{-1}\), we use the analytic expression of the inverse of submatrices \(L_i^{-1}\) from Eq. (72) in Appx. B. The determinant of the matrix \(L_I\), given by,

\[
\det L_I = \prod_{i=0, \ldots, \text{even}}^{N_L} \det L_{i,1} = \prod_{i=0, \ldots, \text{even}}^{N_L} r^{i+1}
\]

(33)

strongly depends on the value of \(r\) at the pupil boundary. The determinant will not form a numerically ill-conditioned problem for model 1 since \(r = 1\) at the boundary. However, this is not the case for model 2 that also uses Eq. (32) but with \(r = \alpha, \alpha < 1\) (see section 5.3.1).

5.2.2 Variational formulation

Model 1 has the same variational terms as model 0, after replacing \(w_0\) with \(w_0^{(2)}\), except for the term in Eq. (19). This term is modified to

\[
R_{\Omega_1} = R_0(w_0^{(1)}, \delta w_0^{(1)}; D_{\Omega_1}^*, \gamma_1, 1, 0)
= \delta A_I^T H_{\Omega_1} A_I
\]

(34)
where $H^{\Omega_1}$ is obtained from $H^{\Omega_q}$ defined in appx. C after extracting only the upper left quadrants from its submatrices $H_{im}^{\Omega_1}$ and setting $r = 1$.

Using Eqs. (20-22), (32) and (34), the linear system of equations to is modified to be

$$\left( T^T_1 H^{\Omega_1} T_1 + R^{\Omega_2}_{k_1 k_2} \right) C = F_{k_2}$$  \hspace{1cm} (35)

where the first term inside the parentheses in the LHS represents the stiffness of subdomain $\Omega_1$ using the new subfunction $w^{(1)}_0$.

5.3 Model 2

Model 2 deals with the lens as a 3-subdomains problem (refer to Fig. 2b). To further improve the model accuracy over model 1 at low DOFs, this model enlarges the membrane area over which the homogeneous solution of the plate equation is used beyond the pupil area. Therefore, it adds a fictitious boundary $\Gamma_{\Omega_{II}}$ that amounts to having a new annular subdomain $\Omega_{III}$. Its displacement ansatz in the lens subdomains becomes

$$\Omega_1: \quad w^{(1)}_0 = A_0^I + B_0^I r^2$$
$$+ \sum_{n=2,4,6,\ldots}^{N_F} \left( A_n^I r^n + B_n^I r^{n+2} \right) \cos(n\theta),$$ \hspace{1cm} (36)

$$\Omega_{II}: \quad w^{(II)}_0 = A_0^{II} + B_0^{II} r^2 + C_0^{II} \ln(r) + D_0^{II} r^2 \ln(r)$$
$$+ \sum_{n=2,4,6,\ldots}^{N_F} \left( A_n^{II} r^n + B_n^{II} r^{n+2} + C_n^{II} r^{-n} + D_n^{II} r^{-n+2} \right) \cos(n\theta),$$ \hspace{1cm} (37)

$$\Omega_{III}: \quad w^{(III)}_0 = \sum_{k=1}^{N_G} C_k \Phi_k(X,Y)$$  \hspace{1cm} (38)
where \( w_0^{(II)} \) is the subfunction of the displacement ansatz over the new annular subdomain \( \Omega_{II} \) and has \( 2N_L \) coefficients to be determined. Its coefficients are \( A_i^{II}, B_i^{II}, C_i^{II} \) and \( D_i^{II} \) where \( i = 0, 2, \cdots N_F \). \( w_0^{(II)} \) uses even terms of the full homogeneous solution to the plate equation including logarithmic and negative-power terms because the subdomain \( \Omega_{II} \) does not enclose the origin. To maximize the membrane area over which \( w_0^{(II)} \) is used, we have chosen the fictitious circle’s ratio \( \gamma_2 = 1 \) in the model-2’s computation in section 5. This choice means that the homogeneous solution is used over the area of the inscribed circle of the square diaphragm. The subfunction \( w_0^{(II)} \) is equivalent to having an annular FEM element similar to.\(^{15}\)

The subfunction \( w_0^{(I)} \) is used for the pupil subdomain as in model 1 while \( w_0^{(III)} \) is used over the subdomain \( \Omega_{III} \) to enforce the clamped conditions, as discussed earlier.

5.3.1 Relationships between coefficients at the boundaries \( \Gamma_{\Omega_I} \) and \( \Gamma_{\Omega_{II}} \)

Model 2 has two internal boundaries \( \Gamma_{\Omega_I} \) and \( \Gamma_{\Omega_{II}} \) at which we will formulate two relationships between coefficients of the subfunctions in adjacent subdomains in a similar way to the procedure followed for model 1.

The first relationship relates coefficients of \( w_0^{(III)} \) and \( w_0^{(II)} \). The radial factor of each trigonometric term of \( w_0^{(II)} \) has four coefficients while those of \( w_0^{(III)} \) have only one coefficient. Thus, to have a one-to-one relation between these two set of coefficients, we will need four boundary conditions at \( \Gamma_{\Omega_{II}} \). Two of them are the continuities of displacement and slope. The other two are the continuities of radial moment \( M_{rr} \) and the vertical shear force \( Q_r \). Since the boundary \( \Gamma_{\Omega_{II}} \) is fictitious and there is no difference in layer structures, we have found that these later continuity conditions are essentially the same as having the \( 2^{nd} \) and the \( 3^{rd} \) derivatives of the displacement continuous (proven in Appx. D). Thus, the relationship can lastly be formulated by setting radial
factors of the trigonometric terms (term by term) of the mechanical quartet \((w_0^{(III)}, w_{0,r}^{(III)}, w_{0,rr}^{(III)}, w_{0,rrr}^{(III)})\) equal to their counterpart radial factors of \((w_0^{(II)}, w_{0,r}^{(II)}, w_{0,rr}^{(II)}, w_{0,rrr}^{(II)})\). This procedure is carried out for the fictitious boundary \(\Gamma_{\Omega_II}\) where \(\gamma_0 = \gamma_2\) and \(r = 1\). In matrix form, the first relationship between coefficients is compactly expressed as

\[
L_{II}|_{r=1}A_{II} = N_{II}C
\]

\[
= \begin{bmatrix}
N_{1,II} & N_{2,II} & \cdots & N_{N_G,II}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_{N_G}
\end{bmatrix}
\] (39)

where

\[
L_{II} = \begin{bmatrix}
L_{0,II} & 0 & 0 & \cdots & 0 \\
0 & L_{2,II} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & L_{N_F,II}
\end{bmatrix}
\] (40)

\[
A_{II} = \begin{bmatrix}
A_{0,II} \\
A_{2,II} \\
\vdots \\
A_{N_F,II}
\end{bmatrix}
\]

\[
A_{i,II} = \begin{bmatrix}
A_i^{II} \\
B_i^{II} \\
C_i^{II} \\
D_i^{II}
\end{bmatrix}
\] (41)
\[
N_{k,II} = \begin{bmatrix}
N_{k,0}^{II} \\
N_{k,2}^{II} \\
\vdots \\
N_{k,N_L}^{II} \\
0
\end{bmatrix},
\]

In Eq. (39), the column vector \(N_{k,II}\) of length \(2N_L\) represents the radial factors and their derivatives for the basis function \(\Phi_k\). The last \(2N_L - 2N_{s,k} - 4\) elements of the \(N_{k,II}\) column are zeros while the first \(2N_{s,k} + 4\) ones can be calculated from Eq. (16). The submatrices \(L_{0,II}, L_{n,II}\) and \(N_{k,n}^{II}\) are given by

\[
L_{0,II} = \begin{bmatrix}
1 & r^2 & \ln(r) & r^2 \ln(r) \\
0 & 2r & 1/r & r(2 \ln(r) + 1) \\
0 & 2 & -1/r^2 & (2 \ln(r) + 3) \\
0 & 0 & 2/r^3 & 2/r
\end{bmatrix},
\]
\[
L_{n,II} = \begin{bmatrix}
    r^n & r^{n+2} \\
    n^1 r^{n-1} (n+2) r^{n+1} \\
    n^2 r^{n-2} (n+2) r^n \\
    n^3 r^{n-3} (n+2) r^{n-1} \\
    r^{-n} & r^{2-n} \\
   (-n)^1 r^{-n-1} (2-n) r^{1-n} \\
   (-n)^2 r^{-n-2} (2-n) r^{-n} \\
   (-n)^3 r^{-n-3} (2-n) r^{-n-1}
\end{bmatrix},
\]

\[N_{k,n}^{II} = \begin{bmatrix}
f_{k,n}(r = 1, \gamma_0 = \gamma_2) \\
f'_{k,n}(r = 1, \gamma_0 = \gamma_2) \\
f''_{k,n}(r = 1, \gamma_0 = \gamma_2) \\
f'''_{k,n}(r = 1, \gamma_0 = \gamma_2)
\end{bmatrix},\]

where \(n^m = n(n-1) \cdots (n-m+1)\).

It is evident that rows of the product \(L_{i,II} A_{i,II}\) and \(\sum_{k=1}^{NG} N_{k,i}^{II} C_k\) respectively correspond to the \(i^{th}\)-radial factor of the trigonometric terms of the quartets \((w_0^{(II)}, w_{0,r}^{(II)}, w_{0,rr}^{(II)}, w_{0,rrr}^{(II)})\) from Eq. (37) and \((w_0^{(III)}, w_{0,r}^{(III)}, w_{0,rr}^{(III)}, w_{0,rrr}^{(III)})\) from Eqs. (16) and (38).

The second relationship relates coefficients of \(w_0^{(I)}\) and \(w_0^{(II)}\). Each trigonometric term of \(w_0^{(II)}\) has four coefficients while those of \(w_0^{(I)}\) have only two coefficients. Thus, to have a one-to-one relation between these two set of coefficients, two conditions are sufficient. The relationship is determined from the continuity of displacement and slope at the boundary \(\Gamma_{\Omega_i}\) where \(r = \alpha = \)
It can be expressed as

\[ \mathbf{L}_I|_{r=\alpha} \mathbf{A}_I = \mathbf{L}^*_{II}|_{r=\alpha} \mathbf{A}_{II} \] (46)

where \( \mathbf{L}_I \) and \( \mathbf{A}_I \) are defined in Eqs. (28) and (29). \( \mathbf{L}^*_{II} \) is obtained from \( \mathbf{L}_{II} \) by extracting only the upper left quadrant of the submatrices \( \mathbf{L}_{i,II} \) since the continuity of displacement and slope are enough to determine the relation between coefficients of \( w_0^{(I)} \) and \( w_0^{(II)} \). Using Eqs. (39) and (46), it is more convenient to express \( \mathbf{A}_{II} \) and \( \mathbf{A}_I \) in terms of the vector \( \mathbf{C} \) as in

\[ \mathbf{A}_{II} = \mathbf{L}^{-1}_{II}|_{r=1} \mathbf{N}_{II} \mathbf{C} = \mathbf{T}_{II} \mathbf{C}, \quad \mathbf{T}_{II} \in \mathbb{R}^{2N_L \times N_G} \] (47)

\[ \mathbf{A}_I = \mathbf{L}^{-1}_I|_{r=\alpha} \mathbf{L}^*_{II}|_{r=\alpha} \mathbf{L}^{-1}_{II}|_{r=1} \mathbf{N}_{II} \mathbf{C} = \mathbf{T}_I \mathbf{C}, \quad \mathbf{T}_I \in \mathbb{R}^{N_L \times 2N_L} \] (48)

where \( \mathbf{T}_{II} \) and \( \mathbf{T}_I \) are model 2’s transformation matrices. Appendix E lists an example of the matrices \( \mathbf{N}_{II}, \mathbf{T}_{II} \) and \( \mathbf{T}_I \) using the first polynomial \( \Phi_1 \). We calculate the transformation matrices using the analytic expression for the inverses. \( \mathbf{L}^{-1}_{II} \) is calculated from Eqs. (85) and (86) in Appx. E while \( \mathbf{L}^{-1}_I \) is calculated in the same procedure as discussed for model 1. The determinant of matrices \( \mathbf{L}_{II} \) given by

\[ \det \mathbf{L}_{II} = \det \mathbf{L}_{0,II} \times \prod_{n=2,\ldots,\text{even}}^{N_F} \det \mathbf{L}_{n,II} \]

\[ = (-16r^{-2}) \prod_{n=2,\ldots,\text{even}}^{N_F} 64(n^4 - n^2)r^{-2} \] (49)

strongly depends on the variables \( n \) and \( r \). The higher the values of \( n \), the more likely that we face
numerically ill-conditioned matrices when solving for the C vector. However, this can be dodged by scaling the equivalent stiffness matrix and the vector C with a diagonal matrix (see section 6.1).

5.3.2 Variational formulation

With three subdomains, the variational formulation of Eq. (2) becomes

$$R_{\Omega I} + R_{\Omega II} + R_{\Omega III} = F_{\Omega I} + F_{\Omega II} + F_{\Omega III}$$

(50)

where

$$R_{\Omega I} = R_o(w_0^{(I)}, \delta w_0^{(I)}; D^*_{\Omega I}, \gamma_2, \alpha, 0)$$

$$= \delta A^T_{\Omega I} H_{\Omega I} A_{\Omega I} = \delta C^T T_{\Omega I} H_{\Omega I} T_{\Omega I} C,$$  \hspace{1cm} (51)

$$R_{\Omega II} = R_o(w_0^{(II)}, \delta w_0^{(II)}; D^*_{\Omega II}, \gamma_2, 1, \alpha)$$

$$= \delta A^T_{\Omega II} H_{\Omega II} A_{\Omega II} = \delta C^T T_{\Omega II} H_{\Omega II} T_{\Omega II} C,$$  \hspace{1cm} (52)

$$R_{\Omega III} = R_{\Box}(w_0^{(III)}, \delta w_0^{(III)}; D^*_{\Omega III})$$

$$- R_o(w_0^{(III)}, \delta w_0^{(III)}; D^*_{\Omega III}, \gamma_2, 1, 0)$$

$$= \delta C^T R_{\Omega III} C,$$  \hspace{1cm} (53)

$$F_{\Omega I} = 0,$$  \hspace{1cm} (54)

$$F_{\Omega II} = R_o(\delta w_0^{(II)}, 1, \alpha) = \delta A^T_{\Omega II} F_{\Omega II} = \delta C^T T_{\Omega II} F_{\Omega II},$$  \hspace{1cm} (55)

$$F_{\Omega III} = R_{\Box}(\delta w_0^{(III)}), - R_o(\delta w_0^{(III)}, 1, 0) = \delta C^T F_{k_2}.$$  \hspace{1cm} (56)

Expressions for the matrices $R_{\Omega I}^{k_1 k_2}, F_{k_2}$ and $F_{\Omega II}$ can be found in Appx. A and C. $H_{\Omega III}$ is obtained from $H_{\Omega q}$ defined in Appx. C by extracting the upper left quadrants of its submatrices $H_{\Omega I}^{\Omega I}$ and $H_{\Omega II}^{\Omega II}$.
setting \( r = \alpha \). \( H^{\Omega_{II}} \) is related to the matrix \( H^{\Omega_{III}} \) by

\[
H^{\Omega_{II}} = H^{\Omega_{III}}|_{r=\alpha} - H^{\Omega_{III}}|_{r=\alpha}.
\] (57)

After collecting the variational terms, we acquire the following linear system of equations

\[
\left( T^T I H^{\Omega_{I}} T_t + T^T_{II} H^{\Omega_{II}} T_{II} + R^{\Omega_{III}}_{k1,k2} \right) C
= T^T_{II} F_{II} + F_{k2}.
\] (58)

Each term inside the parentheses in the LHS of the above equation represents the equivalent stiffness matrix for a subdomain using the specific ansatz for that subdomain. In the same manner, the RHS represents the equivalent forces resulting from each subdomain. However, only two subdomains contribute as they are covered with a piezoelectric film.

6 Comparison between variational models

In this section, we compare the three variational models taking FEM simulations as a reference. We have also carried out a convergence analysis of these models over the displacement and the optical parameters (\( F# \) and RMSWFE).

6.1 Variational solutions versus FEM simulations

In the analyzed lenses, we use material parameters and structural dimensions for the membrane and the piezoelectric stack as specified in.\(^{12}\) We have neglected the effect of platinum and adhesion layers of the piezoelectric stack on the membrane displacement. Moreover, we have only
considered actuation voltage ranges over which the resulting deformation is dominated by bending as opposed to stretching and the equation of the deflection is linear.

The variational solutions of the three models are obtained by solving the three linear systems of equations in Eqs. (24), (35) and (58) for the vector $C$. Enhancing the accuracy of these solutions requires increasing the number of basis functions $N$. However, the higher the $N$ value, the more prone these systems of equations are to become numerically ill-conditioned. That situation can be avoided by multiplying the equivalent stiffness matrix from left and right by diagonal matrices. This diagonal scaling minimizes the euclidean condition number, defined as the ratio of the largest to the smallest singular value of the equivalent stiffness matrix. Thus, we propose a diagonal matrix whose elements are powers of the normalization factor of the Gegenbauer polynomials when their domain extends over the normalized interval $[-1, 1]$. The normalization factor for any function $\phi_{lj}(X,Y)$ is

$$b_{lj} = \frac{\Gamma(\beta)^2}{\pi^{2^{1-2\beta}}} \left[ \frac{ll!l(l+\beta)(j+\beta)}{\Gamma(l+2\beta)\Gamma(j+2\beta)} \right]^{1/2}$$

where $l, j = 0, 2, 4, \cdots$ are indices of the weighted Gegenbauer polynomials. $\beta = 9/2$ and $\Gamma(\cdot)$ is the Gamma function. For pupil opening ratios of interest, we have found that the fourth power is the minimum to dodge numerically ill-conditioning in these linear systems of equations.

Figure 5 shows displacement profiles from variational solutions and FEM simulations for various pupil opening ratios. Models 0 and 1 show similar behavior when pupil opening ratios are small and the Gegenbauer basis is used for most of the membrane area in model 1. Thus, for these ratios, the contribution of the Gegenbauer-basis terms to the electrical enthalpy of model 1 dominates other contributions and the curves resemble those of model 0 which uses this basis only. This
is evident for displacement profiles with $\gamma_1 < 0.6$, as shown in Fig. 5. The larger the value of $\gamma_1$ is, the more dissimilar are the displacement curves of the two models and the better is the agreement between model 1 and FEM.

Model 2 with $N = 3$ has the worst displacement approximation of all models for most pupil openings, but this improves with increasing $N$ value. Model 2 with $N = 7$ provides better displacement approximations than the other models for all pupil openings of interest. In Fig. 5, this becomes particularly clear for the displacement curves of model 2 with $\gamma_1 \leq 0.3$ when compared to models 0 and 1.

To compare the variational solution for $w_{NG}$ of the different models to the FEM result $w_{FEM}$, we monitor the $l_2$ relative error norm given by

$$
\zeta_w = \left[ \frac{\sum (w_{\text{FEM}} - w_{\text{NG}})^2}{\sum w_{\text{FEM}}^2} \right]^{1/2}.
$$

The sums in the above equation are over a set of discrete points of the cross-sectional displacement profiles shown in Fig. 5. $\zeta_w$ curves for different models and various $\gamma_1$ values are shown in Fig. 6. It is evident that model 0 shows decreasing, in most cases, nonmonotonic trends for increasing $N$. However, model 1 shows smoothly decreasing trends and even reaches the highest accuracy at certain $N$ values for $\gamma_1$ ratios $\geq 0.4$.

Model 2’s approximations appear to be worse with lower order $N$, but they improve with increasing $N$. After $N = 7$, the error flatten for all pupil opening ratios of interest and model 2 reaches the highest accuracy. Thus, model 2 can outperform the other models with only 10 DOFs according to Tab. 1.

It is notable that the $\zeta_w$ curves do not converge to zero for any model. This is to be expected
Fig 5 Displacement profiles in $xz$-plane from FEM and different models at $N = 3$ and $N = 7$ for different values of ratio $\gamma_1$ with piezoelectric material at $V_p = -10V$. 
because of the neglect of the inplane displacements. However, this elimination of DOFs that could have been assigned for inplane displacements has simplified the variational formulation significantly.

![Graphs showing relative error norm for the displacement versus number of polynomials N for different models.](image)

**Fig 6** $l_2$ relative error norm for the displacement versus number of polynomials $N$ for different models.

### 6.2 Optical Simulations

To evaluate the lens optical performance, we import the lens displacement from both variational solutions and FEM simulations on a $512 \times 512$ grid. We use ray tracing in an optical-design program (Zemax)\(^4\) to evaluate the lens’ optical parameters such as $F$-number ($F\#$) and RMSWFE. $F\#$ is defined as $f/(\gamma_1 a)$, where $f$ is the focal length defined as the distance in between the lens’
flat surface to the minimum on-axis spot. The on-axis wavefront error (WFE) is the optical path
difference between the constant phase surface of the wave coming out of the lens and a reference
sphere having its center at the image plane and its radius equal to the separating distance between
lens exit pupil and the image plane.\textsuperscript{24}

Figure 7 shows the optical parameters from different models with various $N$ compared to
those from FEM simulations. Model 0’s approximations show oscillatory behavior for the optical
parameters with increasing $N$ similar to the $\zeta_w$ curves. Model 1’s approximations approach the
optical parameters from FEM in a more uniform way. For $\gamma_1 \leq 0.3$, approximations from models
0 and 1 become similar for the same reason mentioned earlier for their displacements.

For model 2 with $N = 1$, $\zeta_w$ is greater than 0.8 for most $\gamma_1$ values of interest. Thus, we have
omitted optical parameters at that particular value of $N$. The optical parameters from Model 2
rapidly approach those of FEM and increasing the value of $N$ above 7 does not add any further
improvements.

To assess the ability of the variational models in approximating the optical parameters over
various pupil openings, we monitor the $l_2$ relative error norms of $1/F\#$ and RMSWFE expressed
as

$$
\zeta_{1/F\#} = \left[ \frac{\sum_{\gamma_1} (1/F\#_{\text{FEM}} - 1/F\#_{\text{Model}})^2}{\sum_{\gamma_1} (1/F\#_{\text{FEM}})^2} \right]^{1/2},
$$

(61)

$$
\zeta_{\text{RMSWFE}} = \left[ \frac{\sum_{\gamma_1} (\text{RMSWFE}_{\text{FEM}} - \text{RMSWFE}_{\text{Model}})^2}{\sum_{\gamma_1} (\text{RMSWFE}_{\text{FEM}})^2} \right]^{1/2}.
$$

(62)

The sums in the above equations are over a set of lenses with $\gamma_1$ values ranging from 0.1 to 0.9
in steps of 0.02. In this aspect, the parameters $\zeta_{1/F\#}$ and $\zeta_{\text{RMSWFE}}$ will indicate the effectiveness
of each variational model to approximate the lens’ optical performance over a wide range of pupil
Fig 7 Reciprocal $F\#$ and RMSWFE versus the ratio $\gamma_1$, all with $V_p = -10V$ and $\lambda = 550nm$ for the three models.
openings. Figure 8 shows these norms versus \( N \) for the three models. Model 0 and 1 show similar behaviors for \( 1/F\# \) curves, but model 1 shows an improved performance for RMSWFE curves. Model-1’s \( \xi_{RMSWFE} \) curve is lower than the one for model 0 by nearly 50% at all \( N \) values. Model 2 starts on the wrong foot, but it becomes more accurate as \( N \) is increased. When \( N \) reaches 7, it becomes the most accurate among the other models. For \( N \geq 7 \), model 2 achieves accuracies of respectively 5.1% and 2.1% for RMSWFE and \( 1/F\# \).

![Fig 8](image)

**Fig 8** \( l_2 \) relative error norm of (a) reciprocal \( F\# \) and (b) RMSWFE versus order \( N \) for three models.

### 7 Discussion

For a certain polynomial order, the three models have the same \( N_G \) DOFs but their accuracy varies depending on the type of basis functions. Model 2 has reduced these DOFs to only 10 as it uses the homogeneous solution of the plate differential equation over most of the membrane area. With the same DOFs, model 0 predicts RMSWFE and \( 1/F\# \) with the respective accuracies of 11.4% and 66.2%, as shown in Fig. 8. It has needed as much as 120 DOFs to bring the accuracies down to those of model 2 in order to provide an acceptable representation of the lens’ optical performance.\(^{12}\)

Models 1 and 2 represent an alternative route to low order models than generic methods such as the MOR methods mentioned in the introduction. Instead of using a a projection approach to
reduce a high-order problem, we have crafted problem-specific trial functions to yield accurate solution starting with a low-order ansatz.

Models 1 and 2 are powerful tools for optical wavefront representations because they can analytically yield the Zernike coefficients to represent the lens surface. Since the displacement ansatz inside the pupil region has the form of Fourier cosine series, it can be easily mapped to Zernike polynomials $Z_{n'}^{m'}$. The squared value of Zernike coefficients is calculated through projection from

**Model 1:**
\[
a_{n'm'}^2 = \int_0^1 \int_0^{2\pi} w_0^{(1)}(r) Z_{n'}^{m'}(r, \theta) r \, dr \, d\theta, \quad (63)
\]

**Model 2:**
\[
a_{n'm'}^2 = \int_0^1 \int_0^{2\pi} w_0^{(I)}(\alpha r) Z_{n'}^{m'}(r, \theta) r \, dr \, d\theta \quad (64)
\]

where

\[
Z_{n'}^{m'} = R_{n'}^{m'}(r) \cos(m'\theta)
\]
\[
= \sum_{k=0}^{n' - m'} \eta_{n'm'k} r^{n' - 2k} \cos(m'\theta), \quad (65)
\]
\[
\eta_{n'm'k} = (-1)^k \binom{n' - k}{k} \binom{n' - 2k}{\frac{n' - m'}{2} - k} \quad (66)
\]

and $n'$ and $m'$ are nonnegative even integers due to the lens symmetry. Their respective maximum values are $N_F + 2$ and $N_F$ from the $w_0^{(1)}$ (or $w_0^{(I)}$) expression and according to the definition of Zernike polynomials. In Eq. (64), the radial variable is scaled by the factor $\alpha$ since Zernike
polynomials are defined on a reference unit circle that is usually taken as the lens pupil.

From Eqs. (65) and (66), and substituting of \( w^{(1)}_0 \) into Eqs. (63) and (64), it equals

\[
a_{n',m'}^2 = \zeta_{m'} \pi \sum_{k=0}^{n'-m'} \eta_{n'm'k} \left( \frac{A_{m'}^1 \tau_{A_{m'}}}{n' - 2k + m' + 2} + \frac{B_{m'}^1 \tau_{B_{m'}}}{n' - 2k + m' + 4} \right)
\]

(67)

where \( A_i^1 \) and \( B_i^1 \) are the coefficients of the displacement ansatz’ subfunction in the pupil area. \( \zeta_{m'} \) is the Neumann factor that equals 2 if \( m' = 0 \) and 1 otherwise. Due to scaling of the radial variable, the correction factors \( \tau \) are defined as

**Model 1:** \( \tau_{A_0} = \tau_{B_0} = \tau_{A_{m'}} = \tau_{B_{m'}} = 1 \), \hspace{1cm} (68)

**Model 2:** \( \tau_{A_0} = 1, \tau_{B_0} = \alpha^2, \tau_{A_{m'}} = \alpha^{m'}, \tau_{B_{m'}} = \alpha^{m' + 2} \). \hspace{1cm} (69)

In that manner, the reduced models can substitute the FEM mechanical simulations of the lens and directly provide the Zernike coefficients representing the lens sag. To be used in optimization, a MATLAB code with a Dynamic Data Exchange (DDE) extension can export the coefficients \( a_{n',m'} \) to optical programs such as Zemax to directly represent the lens sag\(^{24}\) and calculate the required optical performance metrics.

The reduced linear models are necessary foundations for building few-DOFs nonlinear models. Such models will include residual stresses\(^{26,27}\) and geometric nonlinearities\(^{28}\) due to biasing, and involve solving a system of cubic equations of motion.\(^{28,29}\) This system of equations is difficult to be solved semi-analytically with many DOFs because of the tremendous number of integrals to be
calculated for the quadratic and cubic stiffnesses.

8 Conclusions

We have introduced two reduced-order variational models to predict the linear optoelectromechanical performance of piezoelectrically actuated MEMS tunable lenses. The reduction in model order is due to using a piecewise expansion of trial functions that have different forms in the lens’ pupil and the actuator areas. In particular, we used the exact solution for the elastic plate differential equation in the central lens area. The larger this area is, the lower is the number of DOFs needed to reach high accuracy in terms of mechanical and optical parameters. For example, model 2 needs only 10 DOFs to achieve accuracies of respectively 5.1% and 2.1% for RMSWFE and $1/F\#$.

The ansatz subfunction in the pupil region of the presented models can be easily mapped to Zernike polynomials to represent the lens surface. The Zernike coefficients are obtained on closed form in terms of model DOFs. This enables fast computations of lens optical performance which is useful when optimizing lens designs by extensive exploration of dimensions and material parameters.

Acknowledgments

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References


12 M. A. Farghaly, M. N. Akram, and E. Halvorsen, “Modeling framework for piezo-


Appendix A: Variational energy integrals for model 0

The elements in the matrices $R_{k_{1}k_{2}}^{\Omega_{q}}$ ($q = 1, 2$) and $F_{k_{2}}$ are

$$R_{k_{1}k_{2}}^{\Omega_{q}} = \frac{1}{(a/2)^2} \int_{\Omega_{q}} \left\{ D_{11,\Omega_{q}}^{*} \Phi_{k_{1},XX} \Phi_{k_{2},XX} + D_{12,\Omega_{q}}^{*} \Phi_{k_{1},YY} \Phi_{k_{2},XX} + D_{12,\Omega_{q}}^{*} \Phi_{k_{1},XX} \Phi_{k_{2},YY} + D_{22,\Omega_{q}}^{*} \Phi_{k_{1},YY} \Phi_{k_{2},YY} + 4D_{66,\Omega_{q}}^{*} \Phi_{k_{1},XY} \Phi_{k_{2},XY} \right\} dxdy, \quad (70)$$

$$F_{k_{2}} = -V_{p\bar{p}\bar{p}31} \int_{0}^{1} \int_{0}^{2\pi} \nabla^{2} \Phi_{k_{2}} r dr d\theta. \quad (71)$$

A simple way to numerically calculate the above matrix elements is through using a series expansion of Gegenbauer polynomials and integration masks described in details in Refs. 12, 19.

Appendix B: Model 1

The analytic expression of the inverse of sub-matrices inverses $L_{i,1}$ can be expressed as

$$L_{i,1}^{-1} = \begin{bmatrix} (i + 2)r^{-i}/2 & -r^{-1-i}/2 \\ -ir^{-i-2}/2 & r^{-1-i}/2 \end{bmatrix}. \quad (72)$$
The matrices $N_{1,1}$ and $T_{1,1}$ using the first basis function $\Phi_1$ are given by

$$
N_{1,1} = \begin{bmatrix}
3\gamma_1^8/128 - \gamma_1^6/4 + 5\gamma_1^4/4 - 2\gamma_1^2 + 1 \\
\gamma_1^2(3\gamma_1^6 - 24\gamma_1^4 + 80\gamma_1^2 - 64)/16 \\
0 \\
0 \\
-\gamma_1^4(\gamma_1^4 - 8\gamma_1^2 + 8)/32 \\
-\gamma_1^4(\gamma_1^4 - 6\gamma_1^2 + 4)/4 \\
0 \\
0 \\
\gamma_1^8/128 \\
\gamma_1^8/16
\end{bmatrix},
$$

(73)

$$
T_{1,1} = \begin{bmatrix}
t_{1,1} \\
t_{2,1} \\
\cdots \\
t_{10,1}
\end{bmatrix}^T
$$

(74)
where

\[ t_{1,1} = \frac{5}{4} \gamma_1^4 - 2 \gamma_1^2 - \frac{1}{32} \gamma_1^2 (3 \gamma_1^6 - 24 \gamma_1^4 + 80 \gamma_1^2 - 64) \]
\[ - \frac{1}{4} \gamma_1^6 + \frac{3}{128} \gamma_1^8 + 1, \]
\[ t_{2,1} = \frac{1}{32} \gamma_1^2 (3 \gamma_1^6 - 24 \gamma_1^4 + 80 \gamma_1^2 - 64), \]
\[ t_{5,1} = \frac{1}{8} \gamma_1^4 (\gamma_1^4 - 6 \gamma_1^2 + 4) - \frac{3}{32} \gamma_1^4 (\gamma_1^4 - 8 \gamma_1^2 + 8), \]
\[ t_{6,1} = \frac{1}{16} \gamma_1^4 (\gamma_1^4 - 8 \gamma_1^2 + 8) - \frac{1}{8} \gamma_1^4 (\gamma_1^4 - 6 \gamma_1^2 + 4), \]
\[ t_{8,1} = \frac{1}{128} \gamma_1^8, \]
\[ t_{3,1} = t_{4,1} = t_{7,1} = t_{8,1} = t_{10,1} = 0. \]

Appendix C: Variational energy integrals for models 1 and 2

The spring and force variational terms over circular and ring subdomains are given as

\[ R_\circ(w_0, \delta w_0; D_{\Omega_q}^*, \gamma_0, \alpha_H, \alpha_L) = \delta A^T H_{\Omega_q} A, \] (75)
\[ F_\circ(\delta w_0; \alpha_H, \alpha_L) = V_{\bar{p}} \bar{z}_p e_{31} \delta A^T F \] (76)

where

\[ H_{\Omega_q} = \frac{1}{a^2} \left( D_{41, \Omega_q} H^{1h} + D_{12, \Omega_q} (H^{I_2} + H^{I_3}) \right) \]
\[ + D_{22, \Omega_q} H^{I_4} + 4 D_{66, \Omega_q} H^{I_5}, \] (77)
\[ H^m = \begin{bmatrix}
H^m_0 & 0 & 0 & \cdots & 0 \\
0 & H^m_2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & H^m_{NF}
\end{bmatrix}, \]

\[ A = \begin{bmatrix}
A_0 \\
A_2 \\
\vdots \\
A_{NF}
\end{bmatrix}, \quad A_i = \begin{bmatrix}
A_i \\
B_i \\
C_i \\
D_i
\end{bmatrix}, \quad (78) \]

\[ F = \begin{bmatrix}
F_0 \\
F_2 \\
\vdots \\
F_{NF}
\end{bmatrix}. \quad (79) \]

The submatrix 0 is \(4 \times 4\) zero matrix. In Eq. (75), each term \(H^m\) where \(m = 1, 2, \ldots, 5\) represents one of the five integral terms of Eq. (4). The matrices \(H^2\) and \(H^3\) are equal. Expressions for the submatrices \(H^m\) are listed below. To simplify introducing these submatrices, we define two vector-valued functions

\[ \vec{\Psi}_0 = \begin{bmatrix}
1 \\
r^2 \\
\ln(r) \\
r^2 \ln(r)
\end{bmatrix}, \quad \vec{\Psi}_n = \begin{bmatrix}
r^n \\
r^{n+2} \\
r^{-n} \\
r^{-n+2}
\end{bmatrix} \cos(n\theta). \quad (80) \]
\[ H^I = \int_0^r \int_0^{2\pi} \Psi_{n,rr} \tilde{\Psi}^T_{n,rr} r dr d\theta : \]

\[
H_0^I = \begin{bmatrix}
0 & 0 & 0 \\
0 & 4\pi r^2 & -4\pi \ln(r) & 4\pi r^2(\ln(r) + 1) \\
0 & -4\pi \ln(r) & -\pi r^{-2} & -2\pi \ln(r)(\ln(r) + 3) \\
0 & 4\pi r^2(\ln(r) + 1) & -2\pi \ln(r)(\ln(r) + 3) & \pi r^2(8 \ln(r) + 4(\ln(r))^2 + 5)
\end{bmatrix},
\]

\[
H_n^I = \begin{bmatrix}
\frac{2}{3}(n-1)n^2r^{2n-2} & -\frac{3}{4}r^{2n}(-2n^3 - 4n^2 + 2n + 4) & -\frac{\pi}{2}\pi n^2(n^2 - 1) & \pi n \ln(r)(n-1)^2(n-2) \\
-\frac{\pi}{2}r^{2n}(-2n^3 - 4n^2 + 2n + 4) & \frac{\pi}{2}r^{2n+2}(n+1)(n+2)^2 & \pi n \ln(r)(n+1)^2(n+2) & \frac{\pi}{2}r^2(n^4 - 5n^2 + 4) \\
-\frac{\pi}{2\pi^2}n^2(n^2 - 1) & \pi n \ln(r)(n+1)^2(n+2) & -\pi n^2(n+1)/(2r^{2n+2}) & \pi(-n^3 + 2n^2 + n - 2)/(2r^{2n}) \\
\pi n \ln(r)(n-1)^2(n-2) & \frac{\pi}{2}r^2(n^4 - 5n^2 + 4) & \pi(-n^3 + 2n^2 + n - 2)/(2r^{2n}) & -\frac{\pi}{2}r^{2n-2}(n-1)(n-2)^2
\end{bmatrix},
\]

\[ H^I_2 = \int_0^r \int_0^{2\pi} \frac{1}{r} \tilde{\Psi}_{n,r} + \frac{1}{r^2} \tilde{\Psi}_{n,\theta} \tilde{\Psi}^T_{n,rr} r dr d\theta : \]

\[
H_0^I_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 4\pi r^2 & 0 & 2\pi r^2(2\ln(r) + 1) \\
0 & 0 & \pi r^{-2} & 2\pi \ln(r) \\
0 & 2\pi r^2(2\ln(r) + 1) & 2\pi \ln(r) & \pi r^2(8 \ln(r) + 4(\ln(r))^2 + 5)
\end{bmatrix},
\]

\[
H_n^I_2 = \begin{bmatrix}
-\frac{\pi}{2}r^{2n-2}(n-1) & \frac{\pi}{2}r^{2n}(-n^3 + n) & -\pi(-n^4 + n^2)/(2r^2) & -\pi n^2 \ln(r)(n-1)^2 \\
\frac{\pi}{2}r^{2n}(-n^3 + n) & \pi r^{2n+2}(-n^3/2 - n^2/2 + 2n + 2) & -\pi n^2 \ln(r)(n+1)^2 & -\frac{\pi}{2}r^2(n^4 + 3n^2 - 4) \\
-\pi(-n^4 + n^2)/(2r^2) & -\pi n^2 \ln(r)(n+1)^2 & \pi n^2(n+1)/(2r^{2n+2}) & -\pi(-n^3 + n)/(2r^{2n}) \\
-\pi n^2 \ln(r)(n-1)^2 & -\frac{\pi}{2}r^2(n^4 + 3n^2 - 4) & -\pi(-n^3 + n)/(2r^{2n}) & -\pi r^{2n-2}(n-1)^2(2n + 2n - 2)
\end{bmatrix},
\]
\[ H_{I_4}^r = \int_0^r \int_0^{2\pi} \left( \frac{1}{r} \vec{\Psi}_{n,r} + \frac{1}{r^2} \vec{\Psi}_{n,\theta \theta} \right) \left( \frac{1}{r} \vec{\Psi}_{n,r}^T + \frac{1}{r^2} \vec{\Psi}_{n,\theta \theta}^T \right) r \, dr \, d\theta : \]

\[ H_{I_4}^0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 4\pi r^2 & 4\pi \ln(r) & 4\pi r^2 (\ln(r) + 1) \\
0 & 4\pi \ln(r) & -\pi r^{-2} & 2\pi \ln(r) (\ln(r) + 1) \\
0 & 4\pi r^2 (\ln(r) + 1) & 2\pi \ln(r) (\ln(r) + 1) & \pi r^2 (4(\ln(r))^2 + 1)
\end{bmatrix}, \]

\[ H_{I_4}^n = \begin{bmatrix}
\frac{\pi}{2} n^2 r^{2n-2} (n-1) & -\frac{\pi}{4} r^{2n} (-2n^3 + 4n^2 + 2n - 4) & -\frac{\pi}{2} n^2 (n^2 - 1)/r^2 & \pi n \ln(r)(n-1)^2 (n+2) \\
-\frac{\pi}{4} r^{2n} (-2n^3 + 4n^2 + 2n - 4) & \frac{\pi}{2} r^{2n+2} (n+1)(n-2)^2 & \pi n \ln(r)(n+1)^2 (n-2) & \frac{\pi}{2} r^2 (n^4 - 5n^2 + 4) \\
-\frac{\pi}{2} n^2 (n^2 - 1)/r^2 & \pi n \ln(r)(n+1)^2 (n-2) & \pi n \ln(r)(n+1)^2 (n-2) & \pi (-n^3 - 2n^2 + n + 2)/(2r^{2n}) \\
\pi n \ln(r)(n-1)^2 (n+2) & \frac{\pi}{2} r^2 (n^4 - 5n^2 + 4) & \pi (-n^3 - 2n^2 + n + 2)/(2r^{2n}) & -\frac{\pi}{2} r^{2-2n} (n-1)(n+2)^2
\end{bmatrix}, \]

\[ L_4 = \left[ \begin{array}{c}
\frac{\pi}{4} n^2 r^{2n-2} (n-1) \\
\frac{\pi}{4} r^{2n} (-2n^3 + 4n^2 + 2n - 4) \\
\frac{\pi}{4} r^{2n} (-2n^3 + 4n^2 + 2n - 4) \\
\frac{\pi}{4} r^{2n} (-2n^3 + 4n^2 + 2n - 4)
\end{array} \right] \]

\[ H_{I_5}^r = \int_0^r \int_0^{2\pi} \left( \frac{1}{r} \vec{\Psi}_{n,r} - \frac{1}{r^2} \vec{\Psi}_{n,\theta \theta} \right) \left( \frac{1}{r} \vec{\Psi}_{n,r}^T - \frac{1}{r^2} \vec{\Psi}_{n,\theta \theta}^T \right) r \, dr \, d\theta : \]

\[ H_{I_5}^0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \]

\[ H_{I_5}^n = \begin{bmatrix}
2\pi n^2 r^{2n-2} (n-1) & 2\pi n^2 r^{2n} (n^2 - 1) & 2\pi n^2 (n^2 - 1)/r^2 & -4\pi n^2 \ln(r)(n-1)^2 \\
2\pi n^2 r^{2n} (n^2 - 1) & 2\pi n^2 r^{2n+2} (n+1) & -4\pi n^2 \ln(r)(n+1)^2 & -2\pi n^2 r^2 (n^2 - 1) \\
2\pi n^2 (n^2 - 1)/r^2 & -4\pi n^2 \ln(r)(n+1)^2 & -2\pi n^2 (n+1)/r^{2n+2} & -2\pi n (n^2 - 1)/r^{2n} \\
-4\pi n^2 \ln(r)(n-1)^2 & -2\pi n^2 r^2 (n^2 - 1) & -2\pi n (n^2 - 1)/r^{2n} & -2\pi n^2 r^{2-2n} (n-1)
\end{bmatrix}, \]
The dashed lines are to ease visualizing the upper left quadrant of the matrices $H_i^k$ which is used in models 1 and 2. The matrix $F$ and its submatrics are given by

$$F = \int_0^r \int_0^{2\pi} \left( \bar{\Psi}_{n,rr} + \frac{1}{r} \bar{\Psi}_{n,r} + \frac{1}{r^2} \bar{\Psi}_{n,\theta \theta} \right) r \, dr \, d\theta$$

$$F_0 = \begin{bmatrix} 0 \\ 4\pi r^2 \\ 0 \\ 2\pi r^2 (1 + 2 \ln(r)) \end{bmatrix},$$

$$F_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Appendix D: Moment and shear force continuity conditions**

Here we investigate the continuity conditions for the radial moments and shear forces assuming that the continuity conditions over displacement and slope are satisfied. Since there is no difference in layer structures in model 2 at the fictitious boundary, there is no change in the flexural stiffnesses $D_{ij}^\ast$. The radial moment and vertical shear force are given by

$$M_{rr} = -D_{11}^\ast w_{0,rr} + \frac{D_{12}^\ast}{r} (w_{0,r} + w_{0,\theta \theta}),$$

$$Q_r = -D_{11}^\ast \left( \frac{w_{0,rr}}{r} + w_{0,rrr} \right) - (D_{12}^\ast + 2D_{66}^\ast) \frac{w_{0,\theta \theta r}}{r^2} + (D_{12}^\ast + 2D_{66}^\ast + D_{22}^\ast) \frac{w_{0,\theta \theta}}{r^3} + D_{22}^\ast \frac{w_{0,r}}{r^2}.$$
Using Eq. (16), $w_{0,\theta\theta}$ becomes
\begin{equation}
\begin{aligned}
w_{0,\theta\theta} &= - \sum_{n=2,\text{even}}^{N_{s,k}} n^2 f_{k,n}(r, \gamma_0) \cos(n\theta) .
\end{aligned}
\end{equation}

Due to the orthogonality property of the trigonometric functions, the Fourier coefficients (for $n \geq 2$) of $w_{0,\theta\theta}$ are related to those of $w_0$ by
\begin{equation}
f_{k,n}^{w_{0,\theta\theta}} = -n^2 f_{k,n}^{w_0} .
\end{equation}

Thus, at the fictitious boundary $w_{0,\theta\theta}$ is continuous as long as $w_0$ is continuous. From Eq. (81), the only term left to satisfy moment continuity is to enforce a continuity condition on the second derivative of the displacement. In a similar manner, the continuity of the third derivative of the displacement corresponds to the continuity for vertical radial shear force $Q_r$. 

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Appendix E: Model 2

The analytic expression of the inverse of the submatrices $L_{0,\Pi}$ and $L_{n,\Pi}$ can be expressed as

$$L_{0,\Pi}^{-1} = \begin{bmatrix}
1 & -r(\ln(r) + 2)/4 \\
0 & (\ln(r) + 2)/(4r) \\
0 & r/4 \\
0 & -1/(4r)
\end{bmatrix},$$

$$L_{n,\Pi}^{-1} = \begin{bmatrix}
\frac{(n^2-4)r^{-n}}{8(n-1)} & \frac{(n^2+3n-1)r^{1-n}}{8n(n-1)} \\
\frac{-n(n-2)r^{-n-2}}{8(n+1)} & \frac{-(n^2-n+1)r^{-n-1}}{8n(n+1)} \\
\frac{-r^2(n^2-4)}{8(n+1)} & \frac{-r^{n+1}(-n^2+3n+1)}{8n(n+1)} \\
\frac{n(n+2)r^{n-2}}{8(n-1)} & \frac{-r^{n-1}(n^2+n+1)}{8n(n-1)}
\end{bmatrix},$$

(85)

$$L_{n,\Pi}^{-1} = \begin{bmatrix}
\frac{-r^2}{8n} & \frac{-r^3}{8n(n-1)} \\
\frac{r^{-n}}{8n} & \frac{r^{1-n}}{8n(n+1)} \\
\frac{r^{n+2}}{8n} & \frac{-r^{n+3}}{8n(n+1)} \\
\frac{-r^n}{8n} & \frac{r^{n+1}}{8n(n-1)}
\end{bmatrix}.$$
The matrices $N_{1,II}$, $T_{1,II}$ and $T_{1,1}$ using the first base function $\Phi_1$, are given by

$$N_{1,II} = \begin{bmatrix}
3\gamma_0^8/128 - \gamma_0^6/4 + 5\gamma_0^4/4 - 2\gamma_0^2 + 1 \\
\gamma_0^2(3\gamma_0^6 - 24\gamma_0^4 + 80\gamma_0^2 - 64)/16 \\
\gamma_0^2(21\gamma_0^6 - 120\gamma_0^4 + 240\gamma_0^2 - 64)/16 \\
3\gamma_0^4(21\gamma_0^4 - 80\gamma_0^2 + 80)/8 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\gamma_0^4(\gamma_0^4 - 8\gamma_0^2 + 8)/32 \\
-\gamma_0^4(\gamma_0^4 - 6\gamma_0^2 + 4)/4 \\
-\gamma_0^4(7\gamma_0^4 - 30\gamma_0^2 + 12)/4 \\
-3\gamma_0^4(7\gamma_0^4 - 20\gamma_0^2 + 4)/2 \\
0 \\
0 \\
0 \\
0 \\
\gamma_0^8/128 \\
\gamma_0^8/16 \\
7\gamma_0^8/16 \\
21\gamma_0^8/8
\end{bmatrix}, \quad (87)$$
\[
\mathbf{T}_{1,II} = \begin{bmatrix}
243\gamma_2^8/128 - 7\gamma_2^6 + 25\gamma_2^4/4 + 1 \\
-\gamma_2^2(15\gamma_2^6 - 54\gamma_2^4 + 40\gamma_2^2 + 16)/8 \\
\gamma_2^4(27\gamma_2^4 - 96\gamma_2^2 + 80)/16 \\
\gamma_2^4(9\gamma_2^4 - 36\gamma_2^2 + 40)/4 \\
0 \\
0 \\
0 \\
0 \\
\gamma_2^4(5\gamma_2^4 - 16)/64 \\
-\gamma_2^6(3\gamma_2^2 - 8)/32 \\
\gamma_2^8/64 \\
-\gamma_2^8/32 \\
0 \\
0 \\
0 \\
0 \\
\gamma_2^8/128 \\
0 \\
0 \\
0
\end{bmatrix}, \quad (88)
\]

\[
\mathbf{T}_{1,1} = \begin{bmatrix} t_{1,1} & t_{2,1} & \cdots & t_{10,1} \end{bmatrix}^T \quad (89)
\]
where

\[
\begin{align*}
t_{1,1} &= \frac{15}{4} \gamma_2^4 - 4 \gamma_2^6 + \frac{135}{128} \gamma_2^8 \\
&\quad + \alpha^2 (-5 \gamma_2^4 + \frac{9}{2} \gamma_2^6 - \frac{9}{8} \gamma_2^8) \\
&\quad + \ln \alpha (5 \gamma_2^4 - 6 \gamma_2^6 + \frac{27}{16} \gamma_2^8) + 1, \\
t_{2,1} &= \frac{\gamma_2^2}{32 \alpha^2} \left( 80 \gamma_2^2 - 96 \gamma_2^4 + 27 \gamma_2^6 \\
&\quad + \alpha^2 (-64 + 72 \gamma_2^4 - 24 \gamma_2^6) \\
&\quad + \alpha^2 \ln \alpha (320 \gamma_2^2 - 288 \gamma_2^4 + 72 \gamma_2^6) \right), \\
t_{5,1} &= \frac{\gamma_2^2}{64 \alpha^8} (5 \gamma_2^4 \alpha^8 - 8 \gamma_2^4 \alpha^2 + 5 \gamma_2^4 - 16 \alpha^8), \\
t_{6,1} &= -\frac{\gamma_2^2}{32 \alpha^{10}} (3 \gamma_2^2 \alpha^{10} - 3 \gamma_2^2 \alpha^2 + 2 \gamma_2^2 - 8 \alpha^{10}), \\
t_{9,1} &= \frac{1}{128} \gamma_2^8, \\
t_{3,1} &= t_{4,1} = t_{7,1} = t_{8,1} = t_{10,1} = 0.
\end{align*}
\]

Appendix F: Nomenclature

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>\bar{e}_{31}</td>
<td>The effective longitudinal e-form piezoelectric coupling factor.</td>
</tr>
<tr>
<td>\alpha</td>
<td>The ratio ( \gamma_1 / \gamma_2 ).</td>
</tr>
<tr>
<td>\alpha_H</td>
<td>The upper limit of the normalized radius ( r ).</td>
</tr>
<tr>
<td>\alpha_L</td>
<td>The lower limit of the normalized radius ( r ).</td>
</tr>
<tr>
<td>\delta H</td>
<td>The virtual variation of the electrical enthalpy.</td>
</tr>
</tbody>
</table>
$\delta W$ The virtual variation of the potential energy due to external applied forces.

$\delta w_0$ The virtual variation in the displacement $w_0$.

$\Gamma(\cdot)$ The Gamma function.

$\gamma_0$ The ratio of the reference circle diameter to the square plate side $a$.

$\gamma_1$ The ratio of the pupil opening diameter to the square plate side $a$.

$\gamma_2$ The ratio of the fictitious circle diameter to the square plate width $a$.

$\Gamma_{\Omega_i}$ A boundary separate two subdomains.

$\lambda$ light wavelength.

$A_{\Omega_{II}}$ The coefficients vector in domains $\Omega_{II}$ and $\Omega_2$. Its size is $2N_L \times 1$.

$A_{\Omega_I}$ The coefficients vector in domains $\Omega_I$ and $\Omega_1$. Its size is $N_L \times 1$.

$C$ The Gegenabuer coefficient vector of size $N_G \times 1$.

$D^*$ Modified flexural rigidities matrix.

$H^{ii}$ The $ith$ variational integral written in matrix form.

$H^{ii}_{nn}$ The $nth$ submatrix of the $H^{ii}$.

$T_{\Omega_1}$ Transformation matrix between coefficients in the domains $\Omega_1$ and $\Omega_2$ in model 1.

$T_{\Omega_{II}}$ Transformation matrix between coefficients in the domains $\Omega_{II}$ and $\Omega_{III}$ in model 2.

$T_{\Omega_{II}}$ Transformation matrix between coefficients in the domains $\Omega_I$ and $\Omega_{II}$ in model 2.
\( \text{d}n \) The differential length vector normal to the lens’ outer boundary \( \Gamma_\Omega \).

\( \Omega \) The lens’ planar domain.

\( \Omega_i \) A subdomain of the planar domain \( \Omega \).

\( z_p \) The distance between the center of the piezoelectric layer and the reference plane.

\( \Phi_k \) The single-indexed Gegenbauer polynomial in the cartesian form.

\( \phi_{l,i} \) The double-indexed weighted Gegenbauer polynomial in the cartesian form.

\( \tilde{\Phi}_k \) The single-indexed weighted Gegenbauer polynomial in the polar form.

\( \zeta_{\text{w}} \) \( l_2 \) relative error norm for the displacement.

\( \zeta_{1/F\#} \) \( l_2 \) relative error norm for the reciprocal of F-number.

\( \zeta_{\text{RMSWFE}} \) \( l_2 \) relative error norm for the RMSWFE.

\( a \) The side length of the square diaphragm.

\( A_{ii}^I, B_{ii}^I, C_{ii}^I, D_{ii}^I \) coefficients of \( w_0^{(I)} \).

\( A_i^I, B_i^I \) coefficients of \( w_0^{(1)} \) and \( w_0^{(I)} \).

\( b_{ij} \) The normalization factor for polynomial \( \phi_{l,i} \).

\( D_{ij,\Omega_i}^* \) Modified flexural rigidities for the subdomain \( \Omega_i \).

\( D_{ij}^{\text{gl}} \) Flexural rigidities for the glass layer only.

\( D_{ij}^p \) Flexural rigidities for the piezoelectric layer only.
\( f'_{k,n}(r, \gamma_0) \) The derivative of the \( n^{th} \) Fourier coefficient for the \( k^{th} \) Gegenbauer basis function with respect to the variable \( r \).

\( F\# \) F-number.

\( f_{k,n}(r, \gamma_0) \) The \( n^{th} \) Fourier coefficient for the \( k^{th} \) Gegenbauer basis function.

\( N \) The number Gegenbauer polynomials in X or Y directions.

\( n \) Fourier index ranging from 2 to \( N_F \).

\( N_L \) The number of coefficients for the subfunctions \( w_0^{(1)} \).

\( N_F \) The maximum order of Fourier series terms to fully expand set of \( N_G \) weighted Gegenbauer polynomials.

\( N_G \) The number of DOFs for the variational models.

\( N_{s,k} \) The sufficient order of Fourier series terms to fully expand the basis function \( \Phi_k \).

\( N_{s,ij} \) The sufficient order of Fourier series terms to fully expand a doubled-indexed Gegenbauer polynomial \( \phi_{ij} \).

\( r, \theta \) Normalized polar coordinates.

\( RMSWFE \) RMS-wavefront-error.

\( V_p \) The actuation voltage applied across the piezoelectric stack.

\( w_0 \) Displacement in the \( z \)-direction.

\( w_{FEM} \) FEM simulation for the displacement in the \( z \)-direction.
$w_{NG}$  The variational solution for the displacement in the $z$-direction with $N_G$ basis functions.

$x, y$  Cartesian coordinates.

$X, Y$  Normalized Cartesian coordinates.

$C_k$  The Gegenabuer coefficient for the polynomial $\Phi_k$.

$F_\circ$  The force variational integral over an annular (or circular) subdomain.

$F_{\Omega_i}$  The force variational integral over a subdomain $\Omega_i$.

$F_\Box$  The force variational integral over a square cartesian subdomain.

$R_\circ$  The spring variational integral over an annular (or circular) subdomain.

$R_{\Omega_i}$  The spring variational integral over a subdomain $\Omega_i$.

$R_\Box$  The spring variational integral part over a square subdomain.

Mahmoud A. Farghaly is currently a PhD researcher at the University of Southeast Norway. He received his BA and MSc degrees in Electrical Engineering from Assiut University, Egypt in 2010 and 2013, respectively. His current research interests include design and modeling of MEMS sensors and actuators for mobile phones such as: (Lorentz-force and inductive) magnetometers and piezoelectric actuators.

Ulrik Hanke (SM06) received the Siv.Ing. degree and the Dr.Ing. degree in physics from the Norwegian University of Science and Technology (NTNU), Trondheim, Norway, in 1989 and 1994, respectively. After two years as a Post-Doctoral Researcher, he worked for a decade as a Research Scientist with the Department of Communication Technology, SINTEF. Since 2006, he
has been with (University College until 2018) University of Southeast Norway, Horten, where he is currently a Professor of Micro- and Nanosystem Technology. His current main research interest is in the theory, design, and modeling of piezoelectric and RF acoustic micro devices.

Muhammad Nadeem Akram is a professor at the University of Southeast Norway. He obtained Ph.D. degree in photonics from Royal Institute of Technology, Stockholm Sweden in 2005. His research interests are: semiconductor lasers, imaging optics, laser projectors and speckle reduction.

Einar Halvorsen is a professor of micro- and nanotechnology at the University of Southeast Norway. He received the Siv.Ing. degree in physical electronics from the Norwegian Institute of Technology (NTH) in 1991, and the Dr.Ing. degree in physics from the Norwegian University of Science and Technology (NTNU, formerly NTH) in 1996. Since then, he has worked both in academia and the microelectronics industry. His current research interests are in theory, design, and modelling of microelectromechanical devices.

Biographies and photographs of the other authors are not available.

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1. (a) Schematic view showing the tunable lens’s principle of operation; both at rest position when $V_p = 0$ and at focus when $V_p$ is nonzero. (b) Cross-sectional view of the tunable lens showing dimensions. (Adapted with permission from Ref. 12, OSA).
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