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Genesis Principles

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Preface

This article is based on one chapter of my M.S. thesis in mathematics education, concerning the genetic principle. A shorter version of the article will appear in the conference report from the Abel-Fauvel conference, which was held in Kristiansand, June 2002.

I would like to thank Roar Eriksen for helping me translating the text from Norwegian, and I would also like to thank my supervisor, Otto B. Bekken for helping me to re-edit the text. To a large extent, my M.S. thesis was based on work of Gert Schubring, and his work was the very motivation for my thesis. I would like to thank him again, for helpful discussions, kind e-mails and lots of kindness during the work with my thesis. I hope that this article will help to clarify some aspects of the theories in this important field of research in mathematics education.

Contents

The genetic principle is a method of didactics that has been highly misunderstood and misinterpreted. This is perhaps mainly due to the name *genetic*, by many believed to be connected to biology; and also the relations with the *biogenetic law*. To avoid some misunderstandings, genetic here refers to the Greek word *genesi*s, meaning *creation* or *development*. The *biogenetic law* is not to be understood as an educational law, but rather as a pointer to the similarities between individual and historical development of knowledge.

In this article, we will discuss the historical development of genetic principles, and we will see that genesis ideas have played a part in educational theory from the very beginning. We hope that the elaboration of this development will clear up the conceptions, and that you, by learning about the development of these principles and theories, can avoid some of the misunderstandings, and thereby overcome some obstacles related to teaching according to the genetic principle. In reality we are today talking about a web of principles: natural, logical, historical, psychological, cognitive, social, cultural, contextual, situated, ... development of mathematical ideas and concepts.

Original Genetic Principles

We come across various kinds of genetic principles in *Schubring (1978)* who emphasises the distinction between historical and psychological genetic principles, whereas Arnauld is referring to a logical genetic method. Furthermore, we also have Bacon's *natural* method.

The logical genetic method is an expression of a rationalistic philosophy. To Arnauld the method could be expressed as the art of sequencing a series of thoughts in the correct logical order, where the goal is either to discover or establish truth. With his historical genetic method, Clairaut is the first to apply the history of mathematics as a foundation for the learning process (Schubring 1978, p. 41 onwards).

The historical genetic method aims to lead pupils from basic to complex knowledge, in much the same manner as mankind has progressed in the history of mathematics. The aim of the psychological genetic method is to let the pupils rediscover, or reinvent, mathematics by using their own aptitude, through experiences in the surrounding environment. Schubring comments on the relation between the historical and the psychological genetic method:

> Beide gehen von der Vorstellung aus, dass es einen Parallelismus zwischen der Entwicklung des Wissens beim individuellen Erkenntnissubjekt und der historisch-gesellschaftlichen Entwicklung des Wissens gibt. Konzeptionelle Basis hierfür ist - explizit oder implizit - das biogenetische Gesetz (Schubring 1978, p. 192)

Bacon regarded the inductive method as the natural method, as it went from the specific to the universal. In this way it was possible to look into and discover truth (Schubring 1978, p. 21).

Historical Development

The genetic principle focuses on development of concepts. It appears that there is a tendency to be a bit narrow-minded when it comes to this principle. Some researchers show clearly in their published articles that they are of the idea that the principle is inextricably connected to the biogenetic law, and what it deals with is essentially a somewhat modified recapitulation theory. In his publication from 1978 Gert Schubring presents a broader picture and traces the theory almost five centuries back in time.

Scholars differ in opinion as regards the origin of the genetic principle. Felix Klein (1849-1925) is held by many to be the originator, but the didacticians in the nineteenth and early twentieth century appear to have had a deeper historical awareness as regards the roots. Karl Mager claimed that Comenius and Ratke were predecessors of the principle, whereas other sources point out Lindner and Francis Bacon. Clairaut is pointed out by the didactician M. Simon as the first person ever to apply the genetic-heuristic method in the field of geometry (Schubring 1978, p. 16-17).

As we are to look more closely at the historical development, it is only natural to begin with Francis Bacon. He was the earliest of these theoreticians, and even Comenius holds Bacon's work as the basis for his own ideas.

Bacon and Comenius: Early Versions of the Genetic Principle

Francis Bacon (1561-1626) introduced *the natural method of teaching*. He is also considered the founder of the new interpretation of science. Comenius and Ratke based their work on Bacon's studies, and jointly these three are regarded as the predecessors of genetic principles (Schubring 1978, p. 17 onwards). Gradually, the ideas of a natural method of teaching as well as a natural progression in teaching have gained a lot of admirers.

In contrast to his contemporaries: Galilei, Kepler and Descartes, Bacon did not come up with any new scientific discoveries, and he exercised no direct influence on the science of education. However, he had a massive indirect influence as the founder of a new scientific theory, or more specifically the very principle of rationality in science. This new scientific theory proved to be a break with the scholastic theory of the Middle Ages. It developed a method for discovering new knowledge, which we may refer to as the inductive method. Bacon called it a natural method, as it had the *very nature of things* as its goal (Schubring 1978, p. 21).

The method goes from the specific to the general. We might argue that this is exactly the manner in which children learn. First they come across specific cases of various phenomena, later they appreciate the existence of general concepts, which the specific cases form part of. It is this indirect influence that makes Schubring regard Bacon as the founder (Schubring 1978, p. 19).

Bacon felt that teaching should be planned in accordance with nature's own pattern of learning. The very idea that there is a connection between the way children acquire knowledge and the way knowledge has come about in history, is fundamental. Bacon felt that the teacher's task should be to lead his pupils on to the roads of science, in the same way as he himself had arrived there. By these roads of science, Bacon meant:

> …the ancients themselves took the same road as I; that I too, after all my toil and moil, shall probably come at last to one or other of those philosophies that prevailed among the ancients. For they too, in the first stages of their studies, prepared a vast number of examples and particulars, and digested them by subject and specific topics in notebooks, and proceeded from them to compose their philosophies and arts (Bacon 1994, p. 126-127).

When Bacon's method is to be applied in teaching, everyday problems, the so-called specific cases, should be the outset, only later should mathematics be made abstract and theorised. Complete theorems should not be the starting point; instead such theorems should be worked out along the way. In this manner Bacon's method has many parallels to what we refer to as the genetic principle.

Noteworthy of this new scientific way of thinking was that the cognitive subject, as Bacon called it, had to be in activity in relation to the cognitive object. Hence the pupil had to be active in order to acquire knowledge; which is a thought well known in the view of learning as found in the theories of constructivism and reinvention, cf. van Amerom 2002.

After Bacon followed a period of development of philosophical thinking, and philosophy's solution to the problem of cognition in the natural sciences. With this followed reflections about the scientific method (Schubring 1978, p. 23). We will therefore take a closer look at some other philosophers and their possible influence on the development.

Jan Amos Komensky, commonly known as Comenius, lived from 1592 to 1670. He was a Czech philosopher, educationist and poet, and he is widely acknowledged as one of the founders of general educational science through his major work *Didactica Magna* completed in 1657. The basis of his educational science was that all humans are co-creative beings. Despite the fact that all his work was publicly burned, it still had considerable influence on later generations (Comenius 1975, p. 11 onwards). Comenius further elaborates on Bacon's natural method:

> Der Unterricht wird in dem Maasse leicht von Statten gehen, als die Unterrichts-methode der Natur folgt. Alles Natürliche geht von selbst (von Raumer vol. II 1843, s. 55).

Von Raumer gives a closer description of Comenius' natural method:

> Jede Sprache, Wissenschaft, Kunst werde zuerst nach ihren einfachsten Rudimenten gelehrt, dann vollständiger, nach Regeln und Beispielen, hierauf systematisch mit Zuziehung der Anomalien. Man vertheile den Unterricht sorgfältig in Klassen, so dass die untere Klasse der zunächst folgenden obern vollständig vorarbeite; die obere dagegen das in der untern Erlernte befestige. Die Natur ist in stetem Fortschritt begriffen, doch so, dass sie nicht etwa das Frühere aufgibt, indem sie Neues beginnt, vielmehr das früher Begonnene fortsetzt, vermehrt und zur Vollkommenheit führt. Jede Klasse werde in bestimmter Zeit absolviert (Schubring 1978, p. 55-57).

We can sense an embryo of Piaget's theory of stages, and we clearly get the impression that Comenius was preoccupied with the natural development of things in teaching. Von Raumer establishes in his three-volume work *Geschichte der Pädagogik* that Comenius considered Bacon's studies to be the framework for his own work, a view also shared by Schubring (von Raumer vol. II 1843, p. 63 and Schubring 1978, p. 19).

Influence from Philosophers and Theoreticians: Descartes, Hobbes, Spinoza & Leibniz

René Descartes (1596-1650) is widely acknowledged as the founder of the rationalistic philosophy and the deductive method in natural science. He applied the deductive method in order to gain new knowledge, but he did not consider it to be in contrast with the genetic method, as he regarded the objects of knowledge as changeable. Such an understanding of identity between the deductive and the logical genetic method is found with several of the thinkers in this period.

To Descartes and the rationalistic philosophers that succeeded him, the general was seen as the easiest, as opposed to Bacon's view. Descartes, therefore, started with the most general problems. He used the theses that he found to be true in order to discover new theses (Schubring 1978, p. 25). This clearly shows how he considered knowledge to be changeable. Everything changes if you, like many modern theoreticians, imagine humans discovering already fully developed knowledge.

The classic definitions of the deductive and inductive methods, tell us that the inductive method goes from the specific to the general and common, whereas the deductive method goes from the general to the specific. Even in contemporary pedagogy you hear talk about inductive and deductive methods of teaching. It appears that the traditional deductive teaching method, often represented by the classic lecture, is losing ground. Furthermore, the gap has grown between the deductive and the genetic method. The inductive method, which is also found in the heuristic teaching method, is considered to be the one focusing more on the pupils and nowadays appears to have most support. At present, it is this method, and not the deductive method, which is often connected to the genetic method.

Like Bacon, Thomas Hobbes (1588-1679) believed that knowledge requires a complete understanding of the causes of things, or their *causals*. He completed the transition from empiricism to rationalism. To Hobbes too there was a connection between the deductive and the genetic method, but he went a bit further than Descartes, as he claimed that the two methods were in fact identical. To many philosophers the essential question is to discover the inner causes of things. Hobbes held that the only way to grasp the content of a subject was to relive the conditions that had created the content (Schubring 1978, p. 27-28).

At the time, the deductive method was commonly regarded as the most scientific, and Baruch Spinoza (1632-1677) believed that scientific methods had to be deductive. He saw no contrast between deduction and development of concepts. Spinoza held that deduction and genesis coincide in a geometric construction. When a circle is defined as a construction, this is the very prototype of a genetic definition. Such a definition will not explain what a geometric object is like, but rather how it is imagined that the concept must emerge. By means of the genetic definition, Spinoza arrives at the very ultimate goal, namely to join the observable reality with the order of man's mental constructions. The identity between deduction and genesis now match the identity between the causes of things and cognition (Schubring 1978, p. 29 onwards).

The deductive method was the path to cognition for Gottfried Wilhelm Leibniz (1646-1716) as well. He was searching for *the logical principles of knowledge*. It was not enough to him that logic described the formal ties to thinking; it should deal with the very factual content of knowledge. He saw logic and combinatorial analysis as sciences based on arithmetic. Leibniz combined the deductive and the exploratory genetic method. He made comparisons between the pupil and the scientist, and believed it was important to understand the source of the problems, not just focus on the end product and its evidence. Understanding the background was the key, as the remaining could be derived from it. When presenting science, it is important to present it as were it one's own discovery, thus the discovery and the process are highlighted, and not just the end product (Schubring 1978, p. 31-32). These thoughts reappear in the works of later theoreticians, such as Toeplitz, who we will soon focus on in more detail.

Development of mathematics: Arnauld and Clairaut

So far we have looked at how the genetic principle developed in philosophy. With the 17th century came a development of general didactics. As Rousseau later came to see it, education and upbringing became a public responsibility, and didactics became naturally connected with the development of public schools. Ratke, who together with Comenius was one of the founders of the discipline, was the first to use the term *didactics*. However, neither Ratke nor Comenius had their main focus on the teaching of mathematics. The teaching of languages, and the mother tongue in particular, was their main concern (Schubring 1978, p. 36-37).

Towards the middle of the 18th century the teaching of mathematics was gaining ground at both college and university level. The teaching took the form of geometry, based on *the Elements* by Euclid. The ultimate goal was to be able to explain and demonstrate (Schubring 1978, p. 39).

Antoine Arnauld (1612-1694) was an important contributor to further development. He held that the relationship between deduction and genesis had to be differentiated, and he made a distinction between the method for research and the method for presentation. Arnauld named the two methods *analysis* and *synthesis*, where the latter served to present truth in a way that was understandable to others. The presentation had to be done according to the natural order of the phenomena, as the philosophers succeeding Bacon also believed, from the common and simple to the special and complex. His criticism of Euclid's *the Elements* was based on this. He noticed the importance of understanding the causes of things, and this could not be achieved through a *reductio ad absurdum*. He criticised Euclid for not adhering to *the true method*, that is to say to go from the simple and common towards the more complex. Instead Euclid jumbles things together, and presents a mixture of lines, triangles and squares. The first book of *the Elements* starts with the construction of an equilateral triangle, and not until later does he explain how a triangle is made up of three given lines. In this way Arnauld showed how *the Elements* is full of deviations from the natural order (Schubring 1978, p. 40 onwards).

In 1741 Alexis Clairaut (1713-1765) wrote a significant work on basic geometry. In the introduction he described his own method as the genetic method. Contrary to Arnauld's logical genetic method, Clairaut endeavoured to apply the history of mathematics as his basis. Consequently, this is the embryo for the historical genetic method. In many ways, Clairaut's theory resembles the common intuitive understanding of the genetic principle. Clearly visible is the thought of a parallelism between the pupils' development of mathematical ideas and the historical development. He believed that mathematics (geometry) had developed in stages. The first steps had been taken by beginners, and hence should be possible to understand for pupils, being themselves beginners. Clairaut viewed surveying as the origin of geometry, and he therefore gave the pupils the chance to discover the principles of surveying land. His hope was that the pupils in this way should become familiar with exploring, and discovering or reinventing mathematics. He clearly wanted to avoid a way of teaching where the teacher presented a mathematical truth by showing a proof, and then not showing how this had been discovered (Schubring 1978, p. 45-46)

Further development by Lindner and Mager

In the early 1800s there was a fundamental shift in the relationship between science and teaching. As a result of the industrial revolution, the technical development and new groundbreaking thoughts, the educational system became increasingly developed and differentiated. The acquisition of new knowledge became the joint goal of both science and teaching. Hence, a compromise between deduction and the genetic principle could no longer be accepted, as had been the case in the past. The educational system was subject to radical change, and several sweeping school reforms were carried out (Schubring 1978, p. 47 onwards).

We have just discussed what appears to be Clairaut's first rudiment of a historical genetic method. However, Schubring still holds that it was Friedrich Wilhelm Lindner (1779-1851) who first used the genetic principle as a pedagogical idea in a historical genetic way (Schubring 1978, p. 59).

Also Karl Mager refers to Lindner as a pedagogical predecessor of the genetic principle, but little is said about the content of Lindner's theory. A possible reason could be that he only published two shorter works on his method: *de methodo historico-genetica in utroque genere institutionis abhibenda cum altiori tum inferiori* (published 1808 in Leipzig) and *de finibus et praesidiis artis paedagogicae secundum principia doctrinae christianae* (published 1826 in Leipzig). Hegelianism inspired Lindner. He was tutored by Carus who was a strong supporter of the genetic principle in natural philosophy. It is often held that Lindner was led to his methods by Bacon's *Organon*.

Lindner strongly criticised the schools' timetables, which he felt was too tied to a cycle of class-break-class, with 45-minute classes. According to him, the genetic method required stamina, and too frequent change of classes and subjects would only breed distraction. He claimed that the historical part of a presentation was the key to everything that followed (Schubring 1978, p. 59-60). His method was bilateral: historical and genetic. The historical method served to put the different subjects in their correct order; what humans first discovered should be taught first. Mathematics was regarded as the oldest science, and hence should be taught first:

> Sie ist das Element aller übrigen Wissenschaften, sie ist Wissenschaftslehre im eigentlichen Sinne des Wortes und muss unter allen daher zuerst vorgeführt werden (Schubring 1978, p. 61).

The historical method thus deals with the order in which each of the subject areas should be taught. Furthermore, Lindner believed that:

> Alle Theile der vorzutragenden Wissenschaft müssen in eine nothwendige natürliche Causalreihe gebracht (dies nenne ich genetisch, auseinander geboren, erzeugt) … werden (Schubring 1978 refers to Lindner 1808b, p. 84).

Lindner believed that such a way of teaching would arouse and keep the interest of the pupils alive. He also thought that the method could be applied in all subject areas. The teacher could draw the knowledge of this natural order in teaching from the history of science, and from studies of the nature of children, where the development of the natural process of creation repeats itself (Schubring1978, p. 62-63).

Karl Mager developed a more comprehensive understanding of the genetic principle for teaching in schools. He had far-reaching thoughts regarding school politics and school organisation as well as the content and methodology of teaching (Schubring 1978, p. 65). Mager was born 1810 in Solingen. After completing his studies in Paris he moved to Berlin. It was here that he established relations with Diesterweg and other important educationists. He also acquired knowledge of Hegelianism, and became fully devoted to this philosophy. Furthermore, he published his first work on the understanding of the genetic principle: *Wissenschaft der Mathematik nach heuristisch-genetischer Methode. Leistfaden beim Schulunterricht*. He later moved to Zurich, where he in 1846 published his fourth and most elaborate synthesis of his conception of the genetic principle: *Die genetische Methode des schulmässigen Unterrichts in fremden Sprachen und Literaturen*. The chance to put his ideas to test came in 1848 when he became rector of the new Realgymnasium in Eisenach.

Mager introduced a distinction between 'school sciences' and 'technical sciences'. He saw three reasons for doing so:

1. der schulmässige Unterricht ist eine Propädeutik für den wissenschaftlichen Unterricht;

> 2. der schulmässige Unterricht muss eine Auswahl der jeweiligen Wissenschaft behandeln, die zum Aufbau des vollständigen Systems nicht genügt;
>
> 3. “der schulmässige Unterricht hat im Zweck der subjektiven Ausbildung der Schüler das erste, in den Forderungen, welche die Wissenschaft an ihre Bearbeiter macht, aber erst das zweite” (Schubring 1978, p. 78).

He also distinguished between the genetic method as it is applied in research, in scientific presentations and teaching in school (Schubring 1978, p. 88-89). Schubring quotes Mager’s definition of the genetic method for scientific presentations:

> Uns ist die genetische Methode diejenige Entwicklung des Gedankens, welche die Entwicklung des Seins, das erkannt werden soll, schrittweise begleitet und treue spiegelt, so dass beide Gebiete sich decken (Schubring 1978, p. 90)

Mager did not present a general definition of the genetic method for teaching in schools. He held that such a specific designation could only be made within each of the subject areas. In an attempt to solve this problem, he distinguished between pupils in two different stages: before or after the age of 13/14. Mager characterised the method of teaching pupils up until the age of 13/14 as follows:

> Die Lehrform auf der propädeutischen Stufe ist durchaus analytisch, für den Lehrgang fehlt es mir an einem rezipierten Namen: Er ist analytisch und syntetisch, und doch auch beides im gewöhnlichen Sinne nicht; am genausten bezeichnet man ihn vielleicht, wenn man ihn in Beziehung auf den Gegenstand kombinatorisch, in Beziehung auf das lernende Subjekt psychologisch-genetisch nennt (Schubring 1978, p. 93).

The real genetic method does not become applicable until the second stage, where the development of scientific material is used in a historical genetic method. In the teaching of mathematics, Mager confined the use of the genetic principle to the field of arithmetic.

Mager’s genetic method became very influential and was praised in periodicals (Schubring 1978, p. 103 onwards). His method was often used in textbooks when presenting for example geometry.

The Understanding of Mathematics Education

As mathematics education gradually emerged as a separate subject area, the genetic principle soon assumed the role as the pivotal didactic principle in German upper secondary schools, the so-called *Gymnasium*. One of the earliest periodicals on mathematics education, *Zeitschrift für den matematischen und naturwissenschaftlichen Unterricht*, soon pointed out the genetic method as the best way of teaching (Schubring 1978, p. 127 onwards).

Discussions about the correct understanding of the concept went high, and there were several competing interpretations. Holzmueller questioned the very use of the word *genetic*, as he interpreted this as being derived from genesis, or the coming into existence of something (Schubring 1978, p. 130-131). Schubring comments:

> Implizit wird hier die Vermischung des ursprünglich philosophisch begründeten Ansatzes von der Beziehung von Allgemeinen und Besonderem und der psychologischen Verankerung der geometrischen Begriffe in der Alltagsanschauung mittels der induktiven Methode deutlich (Schubring 1978, p. 131).

At the time in Germany, when the genetic method was recommended as the best teaching method in mathematics education handbooks, there existed a division between a psychological and a more historical genetic trend. The historical genetic method was founded on the biogenetic law. The Euclidian method was seen as distinct from the genetic (Schubring 1978, p. 132).

Reidt saw the correct way of teaching as being trilateral. The form of instruction should be *heuristic*; the treatment of the factual content *analytical*; and the structural system should be *genetic* (Schubring 1978, p. 134).

Felix Klein and the Genetic Principle

Felix Klein (1849-1925) developed mathematics in a sequence of common sense, with constant references to history (Burn 2000, p. 1). Wittmann even states:

> Die genetische Methode wurde in bezug auf den Mathematik-Unterricht explizit zum ersten Mal von Felix Klein artikuliert (quoted in Schubring 1978, p. 6).

Towards the end of his career Klein was mostly occupied with didactical or organisational issues. His starting point was a desire to create unity within science, a unity more than anything between the theoretical and practical aspects of science. He wanted to bridge the gap between the mathematics taught in primary/secondary schools and the mathematics taught at university level. He wanted to accomplish this by introducing differential and integral calculus at the upper secondary level, as well as by a fundamental restructuring of the whole idea of instruction. At this point he applied the same method as in his *Erlanger Programm*, namely to develop the entire instruction around one main idea. This main idea should now be *the function concept* (Schubring 1978, p. 140-141).

Klein was not content with the way mathematics was presented by his contemporaries. In the early 1900s he published several articles aimed at teachers. In his articles he applied the history in order to present mathematics, as well as to structure the pedagogy for the teacher. He presented a modified version of traditional pedagogy based on four key elements, as we will soon see. The most important element was history.

To Klein, it went without saying that the presentation of any educational material had to be made in accordance with the genetic principle; consequently he never published an introduction to his own interpretation of the genetic principle. Klein used several of the already existing interpretations; both the psychological as well as the historical genetic versions, and referred his work to the biogenetic law (Schubring 1978, p. 142 onwards).

In his book *Elementary Mathematics – from an Advanced Standpoint*, Klein endeavours to give an introduction to arithmetic, algebra and analysis. He starts by presenting how to teach pupils numbers, the very basis of all arithmetic. Speaking on this, he says:

> The manner of instruction as it is carried on in this field in Germany can perhaps best be described by the words intuitive and genetic, i.e. the entire structure is gradually erected on the basis of familiar, concrete things, in marked contrast to the customary logical and systematic method at the university (Klein 1945, p. 6).

He starts with the intuitive and goes gradually onwards. Having done numbers he proceeds to elementary multiplication with small numbers, which he feels that pupils must know by heart.

Next, the pupils are to learn how to multiply numbers with more than one decimal. From the very outset the pupils learn numbers from their everyday applications and from examples. Klein therefore holds that not only terms such as intuitive and genetic should be used in teaching, but also applications (Klein 1945, p. 7).

It is a common argument that mathematics can and should be taught deductively; by starting with certain axioms and by manner of logic deducting everything from there. On this, Klein comments:

> This method, which some seek to maintain upon the authority of Euclid, certainly does not correspond to the historical development of mathematics. In fact, mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upwards. Just so… mathematics began its develop- ment from a certain standpoint corresponding to normal human understanding, and has progressed, from that point, according to the demands of science itself and of the then prevailing interests, now in the one direction toward new knowledge, now in the other through the study of fundamental principles (Klein 1945, p. 15).

The understanding of foundational principles is constantly changing, according to Klein, and there is no end, and hence no initial starting point that could provide an absolute fundament.

After providing an historical outline of the development of logarithms and exponential functions, Klein expresses his complaint on the way this modern development has developed without affecting teaching. He establishes as a fact that teaching and development of mathematics as a branch of knowledge lost all contact with each other in the early 1800s. Based on this, he holds that Euler's definitions and notations have remained in schools, whereas universities to a larger extent have been able to keep up with the development. The result is that what is taught in schools will not be further elaborated at the university level. At the same time universities continue to build their own systems, and face students with the frustrating and often incorrect comment: *you should know this already from school* (Klein 1945, p. 155).

Towards the end of his book Klein sums up his view: There are four elements for a teacher in mathematics to emphasise, and this constitutes the difference between his presentation of material and the customary presentation in textbooks (Klein 1945, p. 236):

1. diagrams illustrating abstract relations
2. emphasis on the connection to related fields
3. emphasis on the historical development
4. examples from popular literature in order to demonstrate the differences between notations used by laymen and notations used by experienced mathematicians

And he goes on to say that:

> If you lack orientation, if you are not well informed concerning the intuitive elements of mathematics as well as the vital relations with neighbouring fields, if, above all, you do not know the historical development, your footing will be very insecure (Klein 1945, p. 236).

Furthermore, from 1898 Klein edited *Encyklopädie der matematischen Wissenschaften mit Einschluss ihrer Anwendungen*. This book provided one of the most complete presentations of mathematics in the 19th century, and it contains several historical footnotes, in order to make the historical origin of the subject areas clear to the reader.

Benchara Branford

An approach between didactics and history of mathematics has long been widely acclaimed, as we have seen, in the German tradition. However, in 1908 when Branford published his *A study of Mathematical Education*, he represented something new in the English speaking culture. Our edition of this work is from 1924.

The biogenetic law is pivotal in Branford's work, and we will look at his interpretation and application of it in practice. A diagram in the beginning of his book points out the relation between the development of mathematical skills in the individual and the development of mathematics historically. The diagram expresses a way of thinking ever-present in his work, and also reappears in his point of view as regards teaching. He held the biogenetic law as a biological theory to be true:

> We can practically take it now as established by a large number of lines of evidence coming through many sciences that the individual does recapitulate in his own development the essential lines through which the race has passed - I say essential lines, not the details. It is, I believe, admitted by experts to be true biologically: it was first found in biology; and now it is seen to be true also for the mental or psychic organization (Branford 1924, p. 47).

Schubring elaborates on Branford's point of view of the development of knowledge:

> Branford geht davon aus, dass das Problem der Wissensentwicklung beim Schüler nicht gelöst ist durch die Aufstellung eines didaktischen Prinzips wie des biogenetischen Grundgesetzes, sondern dass sich damit überhaupt erst die Aufgabe für die Mathematik-Didaktik stellt, theoretische Arbeit und die Erfahrungen und Unterrichtsexperimente der Lehrer systematisch aufeinander zu beziehen (Schubring 1978, p. 300).

Although Branford actively applies the biogenetic law, he has no desire to prove the existence of any essential parallelism between for instance the development of geometrical knowledge in mankind and in the individual (Schubring 1978, p. 304 and Branford 1924, p. 327). Branford's aim is:

> …to show that, for educational purposes, the most effective presentation of geometry to youth, both as regards matter and spirit, is that which, in main outlines, follows the order of the historical evolution of the science (Branford 1924, p. 327).

According to Schubring (1978, p. 305) Branford applies the biogenetic law in three different ways:

- as a foundation for his understanding of conceptual development,
- as a foundation for his understanding of the development of science
- in a narrower sense, as means to developing curricula

As a background for his research, Branford had behind him years of experience as a teacher and, naturally, he had his own understanding of what the teacher's role should be. In accordance with the biogenetic law, the role of the teacher is to structure the teaching in a manner corresponding to the lines suggested by the development of mathematical knowledge in mankind. The teacher's job is to avoid the mistakes made in history. Hence, it is paramount for a teacher to be aware of the history, in order to take possible shortcuts or shorter stops en route, where the aptitude of the pupil should indicate the need to do so (Branford 1924, p.244)

Branford provides numerous examples from his lessons. Among the first observations he presents, is something many teachers have experienced when trying to teach geometry. First of all he thinks that children have an advanced ability to use ideas of solid geometry in practical situations, at the same time as they find it hard to assimilate the formal use of these concepts. Then he goes on to look at the ideas of solid geometry that pupils take with them from their everyday lives into schools. Based on this, he establishes that the history of mathematics is simply one long, continuous development of mathematical concepts. This *law*, as he calls it, should be applied when teaching for instance geometry. We should therefore treat our pupils as brave young pioneers, and their unfinished definitions and assertions should be met with respect and the mild criticism that is due discoverers of such concepts (Branford 1924, p. 11). The pupils should work their way through mathematics in the historically correct order, and start with the geometry of the Greeks.

Branford holds the existence of mathematical knowledge as one of the prerequisites for advances in all sciences and arts. In the same way as mathematics has influenced the progress of the other sciences as well as arts and industry, in return these have had an effect on the development of mathematics. The knowledge of mathematics has grown under the influence of several impulses. Branford makes this distinction (p. 221 onwards):

- Practical impulses: At all times the need to solve everyday problems has contributed to the advance of mathematics. One prime example is how the shepherds of Antiquity carved out their staffs in order to keep count of their animals, and this is widely held to be the coming into existence of numbers and counting.
- Scientific impulses: One example is the enormous progress made in mathematics as a result of the advances in astronomy.
- Aesthetic impulses: This impulse makes us want to study mathematics for its own sake, because we (as mathematicians) see the beauty in it, in much the same way as painters see the beauty in a painting and musicians in a piece of music.

Branford consequently assumes a close relationship between the historical advances of mathematics and the growth of mathematical knowledge in the individual. In much the same way as the historical development of mathematics is influenced by several factors, the individual's development of mathematical knowledge is influenced by different exterior and interior factors.

First of all we notice the influence of the physical environment man was subject to in times past. This is also found in the close connection between the anatomy of the human body and the development of number systems. It is no coincidence that all nations use 5, 10 or 20 as the basis for their numeral systems. There are five fingers on one hand, ten fingers in total, and the number of fingers and toes altogether amounts to twenty.

Secondly, the increase of mathematical ideas has been influenced by what humans have been doing. A number of professions have come into being since the initial hunting and fishing of earlier ages, and each profession has brought its own devices for counting, measuring, weighing and estimating the value of objects. Influence from astronomy and the natural sciences are seen here too.

As regards the individual's expansion of mathematical ideas, it has also been influenced by internal factors. We find here the same aesthetic factors as in the historical development of mathematics (Branford 1924, p. 229 onwards).

Branford draws up an historical sketch of early arithmetic, in order to find out how it can be used as an integral part of teaching. He does this pursuant to the principle of a parallelism between the evolution of the individual and mankind. Historians describe the expansion of the various sciences and the expansion of man's knowledge in the different cultures. The job of discovering the growth of knowledge in children, youths and grown ups is left to psychologists, teachers and possibly parents (Branford 1924, p. 47).

According to Branford children are born with several mental ideas. These ideas, however, can be hard to discern at first. Children have innate ideas about several different mathematical concepts, but they are not, and will never be perfect, Branford holds. A perfect understanding of a concept is impossible to achieve, as long as the words relate to meanings that are contextual (Branford 1924, p. 48).

As many psychologists and other theoreticians have done later, Branford divides the growth of arithmetic ideas and symbolism into stages. He presents five such stages, ranging from the obscure stage, where the child develops an idea of itself in relation to the surrounding world, to a highly advanced stage, where the child performs basic arithmetic operations with numerals. These stages are repeated in the child's further development of algebraic ideas (Branford 1924, p. 49 onwards).

> For there is every reason to believe, looking to the practically unchanged constitution of the human mind for at least several thousands of years back, that those factors which have been throughout essential to the growth of mathematical knowledge in the minds of our ancestors must be closely similar to, if not actually identical in kind with, the main factors that underlie efficient mathematical education in kindergarten, school, and college (Branford 1924, p. 225).

Based on this, Branford translates the fundamental laws of man's development of mathematics into the individual level. His ideas are formulated in a thesis:

> The path of most effective development of knowledge and power in the individual, coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power (Branford 1924, p. 244).

Branford now uses the historical sketch from the beginning of his book to find the most suitable way for the child to assimilate mathematics (Branford 1924, p. 245).

Towards the end of his study Branford discusses the relationship between teaching principles and practice:

> All principles, I take it, represents but partial aspects of reality. Nothing, perhaps, is more fatal to progress and to success in teaching than the attitude of the doctrinary belief in the universal validity of any abstract principle or system of principles, and consequent insistent adherence to it in practice. Principles thus viewed and applied are life-killing mechanisms (Branford 1924, p. 345).

Otto Toeplitz

Whereas Branford focused on school and its mathematics, Otto Toeplitz (1881-1940) had his focus on higher education. Albeit Toeplitz never made a compilation of his understanding of mathematics education, his point of view can be reconstructed to a certain extent by looking at his books and articles. In general, he considers the relationship between mathematics as science and as the object of teaching, from a mathematician's point of view.

Toeplitz felt that in order to understand a concept, an imparting of knowledge (meta knowledge) about the particular concept was required. Therefore, he highlighted a presentation of the material in its conceptual context, and he distinguished between a direct and an indirect genetic method, according to *Schubring 1988*:

> Wenn man die Wurzeln der Begriffe zurückginge, würde der Staub der Zeiten ... von ihnen abfallen, und sie würden wieder als lebensvolle Wesen vor uns erstehen. Und von da aus würde sich dann ein doppelter Weg in die Praxis darbieten: Entweder man könnte den Studenten direkt die Entdeckung in ihrer ganzen Dramatik vorführen und solcherart die Fragestellungen, Begriffe und Tatsachen vor ihnen entstehen lassen - und das würde ich die direkte genetische methode nennen - oder man könnte für sich selbst aus solcher historischen Analyse lernen, was der eigentliche Sinn, der wirkliche Kern jedes Begriffes ist, und könnte daraus Folgerungen für das Lehren dieses Begriffes ziehen, die als solche nichts mehr mit der Historie zu tun haben - die indirekte genetische Methode (Toeplitz 1927, s. 92f.).

When applying the indirect genetic method, there is no need to teach history. The application of this method does not necessarily have anything to do with history, and Toeplitz was not interested in history as such. What mattered to him, and to others who make use of this method, was the very genesis of the concepts. The teacher should follow the genetic path, in much the same way as mankind has gradually progressed from basic to more complex patterns in the course of history.

In 1963, Toeplitz' *"The Calculus – a genetic approach"* appeared in English. The original had been published by Köthe in German in 1949. By writing the book he desired to provide teachers with a model for his indirect genetic method, where he could present the basics of conceptual development. Toeplitz was never able to finish the book, but in the manuscripts left behind he covered the development of analysis up until Newton and Leibniz. The book begins with Zeno's paradoxes about infinite quantities and continues via Pythagoras and Euclid through the history of mathematics. Ergo, he starts with the Greeks' concepts of infinite quantities and concludes with differential and integral calculus. The historical sequence of the concepts is important for the teaching, and in his book Toeplitz presents the definite integral before differential calculus. This is based on the fact that it was the early Greeks who discovered the integral. The only thing left out by the Greeks was Leibniz's integration sign. Hence it was only natural to present the definite integral before differential and integral calculus, which in time did not appear until Newton and Leibniz. Towards the end of his book he goes through the laws of Kepler. He constantly uses history in his approach to the mathematical theories.

Modern expressions of the Genetic Principle

In 1977, Harold M. Edwards published his book *Fermat's Last Theorem – A genetic introduction to algebraic number theory*. He defines the genetic method as the explanation or the assessment of an object or an incident with regards to its origin and development, and he goes on to explain the importance of differentiating between the genetic method and history. History aims to provide a correct picture of the people, ideas or incidents that have influenced the progress of a certain subject. The genetic method, on the other hand, has its main focus on the subject and tries to explain and assess it. History rarely has room for detailed descriptions of theory, whereas the genetic method has no room for detailed studies of events, unless they contribute to increased understanding of the subject (Edwards 1977, p. vi). The genetic method ignores dead ends and mistakes, and focuses on the things that have contributed positively to advancing the theory.

Edwards refers to how Toeplitz described the method:

> …the essence of the genetic method is to look to the historical origins of an idea in order to find the best way to motivate it, to study the context in which the originator of the idea was working in order to find the "burning question" which he was striving to answer (Edwards 1977, p. vii).

This is in contrast to the regular method, where no concern is given to the questions and solely the answers and the fully completed theorems are presented. Edwards has experienced that Toeplitz's method is superior when it comes to overcoming difficulties in learning abstract mathematical theories:

> From a logical point of view only the answers are needed, but from a psychological point of view, learning the answers without knowing the questions is so difficult that it is almost impossible (Edwards 1977, p. vii).

A modern use of the genetic method is also described in the latest ICMI study (Fauvel & van Maanen 2000). Here Kronfellner elaborates on how to apply the indirect genetic method when teaching calculus. He holds that starting with a definition of limits is not a good idea, but he rather recommends using an intuitive idea about an indefinite approach. This method, which he refers to as genetic, or indirect genetic according to Toeplitz, involves no need to mention historical details explicitly. The historical development functions merely as a guiding light, by showing the teacher or the textbook writer the way forward. One argument is that those aspects of a concept that historically has been discovered and applied first, is probably best suited early in the teaching process. In many respects, this method is similar to that of Edwards. The outset is the original problems and then you work your way towards the modern concepts (Fauvel & van Maanen 2000, p. 71).

Another argument given by Kronfellner, is that history tells us how the development of mathematical concepts has taken time. A genetic approach might therefore tell the teacher not to present too complicated mathematical concepts too early in the teaching process (Fauvel & van Maanen 2000, p. 73).

As we have seen already, the genetic principle in its widest sense is a highly complex concept. When the genetic principle is discussed here, it is regarded a teaching principle, as we have seen it evolve in history, before eventually it was formulated by Klein, Branford and Toeplitz. Toeplitz explained the genetic principle in this way:

> Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g., the mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised "Why so?" or "How does one arrive at them?" Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut and dried facts and instead take on fresh and vibrant life again (quoted in Furinghetti & Radford 2000, p. 15).

It is a common belief that mathematics should be presented in a cultural and historical context. Mathematics cut loose from its roots and cultural background has been called *fast food mathematics* (Dennis 2000, p. 802). One possible way of dealing with such fast food mathematics is to apply the genetic principle. A great many scientists seem to be positive when it comes to the principles behind the genetic method (i.e. Burn 1999, Furinghetti & Radford 2000, Selter 1997 and Steiner 1988), and a lot of teachers seem to find parallels to history when teaching mathematics. Furthermore, it seems that teachers with background knowledge of the history of mathematics often see this as an advantage when it comes to their own teaching. Frequently, negative attitudes seem to be based on a slight misunderstanding of the very concept.

The Norwegian Tradition

Some might argue that it is slightly far-fetched to talk about a Norwegian tradition for the genetic principle as a teaching method, due to the very fact that mathematics education only recently became the subject of research in Norway. Even so, Norwegian academics have produced some articles on the genetic principle, and as I consider myself to be part of this tradition, I will include some of it.

Bekken et al. 1978, p. 32 distinguishes between four ways of applying the history of mathematics in a pedagogical context:

1. To follow the course of history in mathematics course.
2. To use historical examples as a "treasure box", to illustrate mor abstract issues.
3. To enlighten the students by telling them about the historical "battle" with the ideas, so that they see that mathematics is not a finished and set theory, but rather constantly evolving.
4. To point at causes behind and the effects of the development of mathematical theories. The causes might be found within mathematics as well as outside of it, and such a presentation of an issue might contribute to an understanding of the nature of mathematics.

All these four points are based on using a genetic approach when teaching mathematics, with an emphasis on the development of ideas. Such an approach has been used in some courses in the past at the Agder University College, Norway.

In August 1988, there was an international conference in Kristiansand towards the end of the ICME-6. This conference was dedicated to the history of mathematics, and organised by Otto B. Bekken of the Agder University College and Bengt Johansson of the University of Gothenburg. The outcome of the conference was presented in the book, *Learn from the Masters*, which deals with several ways of applying the history of mathematics in teaching. The genetic principle is not dealt with explicitly in any of the articles included in the book.

Robert P. Burn is, to be correct, not Norwegian, but as a former professor in mathematics education at the Agder University College, he belongs in many respects to the Norwegian tradition. He says:

> Using history to locate steps in development is the 'genetic method' (Burn 1999, p. 7).

He refers to Toeplitz's definition of the genetic principle, and much like Toeplitz, he claims:

> While the genetic method depends on careful historical scholarship it is not itself the study of history. For it is selective in its choice of history, and it uses a modern symbolism and terminology (Burn 1999, p. 8).

Tone Bulien deals with several ways of applying history when teaching algebra. She elaborates on how to teach algebra using the biogenetic law as well as the genetic method. Branford is highlighted as the primary advocate of a comparison between the historical development of mathematics and pupils' understanding of mathematics. She also points out Klein and Pólya as advocates of such ideas. As regards the genetic method, she refers to Toeplitz. This forms part of the background for her own study, where she uses historical problems actively in teacher training. In her conclusion, she states:

> ... there are many ways to use the history of mathematics in the classroom; the history itself is a rich source of problems and stories. And looking at the different forms of mathematical expressions one might find some sources for a genetic approach or even in some instances that ontogeny recapitulates phylogeny though the opposite was stated in the problem set presented to the student teachers (Bulien 2000, p. 79).

In an article on the development of algebra, Otto B. Bekken presents the genetic principle in this way:

> To me, the genetic method was central, as I had discovered, throughout almost 20 years of teaching, that ideas, concepts and methods that caused problems for the students, also often had a problematic development in the course of history (Bekken 2000, p. 85).

He then refers to the wide variety of literature concerning these theories, with *Schubring 1978* as the starting point. Here Bekken goes on to highlight Vygotsky as a more unknown supporter of the genetic principle. As a guide to teaching, Bekken states:

> In our teaching we should therefore strive to make connections between historical genesis, cognitive/psychological geneisis and logical genesis of mathematical ideas (Bekken 2000, p. 86).

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