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Kristian Staff

Almost all entangled states of two particles exhibit non-locality without inequalities

Supervisor: Leon Loveridge

Abstract

This thesis investigates the constraints put on hidden-variables models by Hardy's argument. We provide a detailed derivation of three different presentations of Hardy's argument and demonstrate that a class of hidden-variables models cannot reproduce the predictions calculated from the mathematical formalism of quantum mechanics. This demonstration utilises a valuation function that, under mild criteria, fails to assign pre-existing values to certain physical quantities in specified pure quantum states. We elaborate upon and clarify the distinction between the quantum mechanical framework and the hidden-variables model, which has been left implicit by other authors. We address the limitations of the argument when applied to maximally entangled states, and substantiate the assertion that the argument remains valid for all pure entangled states, excluding those that are maximally entangled.

1 Introduction

The objective of this thesis is to provide a detailed derivation of the argument known as 'Hardy's Paradox' – originally presented by Lucien Hardy and subsequently reformulated by Sheldon Goldstein – while elucidating certain aspects and limitations of the argument previously left implicit by other authors. Hardy's argument is a thought experiment which provides insight into the non-classical aspects of quantum mechanics, and it highlights the conditions that hidden-variables theories must meet to match the predictions of quantum mechanics, which experimental results have consistently supported.

Hidden-variables models became a field of interest in the early years of quantum theory and sought to replace the probabilistic description of quantum theory with a deterministic theory. The debate about quantum theory intensified following the publication of a paper by Einstein, Podolsky, and Rosen, in which they argued that the quantum theoretical description of reality was incomplete [1]. The paper has had a significant impact on subsequent discussions and research, as well as on the interpretation of quantum theory. In the paper, the authors makes the crucial assumption that a measurement on one part of a system cannot affect the outcome of a measurement on a different system which is spatially separated, a principle which subsequently became known as *locality*. Furthermore, the authors assume a kind of *realism* which satisfies a *criterion of reality* which they define. The authors proposed resolution to the apparent incompleteness of quantum theory was that there existed some additional *elements of reality* which determine the outcome of measurements. These *elements of reality* is often referred to as hidden-variables. However, the authors left open the question of the details of such a theory.

The debate about hidden-variables models was substantially influenced by John Stewart Bell, who in 1964 proposed an empirical test of *local realism*, known as Bell's Theorem [2]. The empirical test relies on an inequality which is violated by quantum theory, but should hold for local hidden-variables models. As Bell himself said: "If [a hidden-variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local." [3]. One such Bell inequality was proposed by John Clauser, Michael Horne, Abner Shimony, and Richard Holt in 1969, known as the CHSH inequality [4]. Subsequent experiments have shown that the inequality is violated, thereby lending credence to quantum theory. Furthermore, Bell contributed in showing the Kochen-Specker theorem which highlights that properties of quantum systems are inherently dependent on the context in which they are measured [5]. It shows that there is no consistent way of assigning pre-existing values to the properties of some quantum systems. Further work on contextuality has been done by David Mermin, who showed a state-independent contextuality argument with two qubits [6]. An experiment proposed by Daniel M. Greenberger, Michael A. Horne, and Anton Zeilinger, known as the GHZ experiment, provides an empirical test of *local realism* in the same manner as Bell's inequality. However, unlike Bell's, this experiment does not rely on inequalities [7]. The results of experiments agree with the predictions of quantum mechanics, and in 2022 Anton Zeilinger won a shared Nobel prize for his contributions to experiments with entangled photons, which established the violation of Bell inequalities.

Similarly to the GHZ experiment, Hardy's argument does not rely on inequalities. However, unlike the GHZ experiment, Hardy's argument does not apply to all iterations of the experiment. In this thesis we will present the argument put forward by Lucian Hardy in his paper "Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Realistic Theories" [8] and his work in extending the argument to almost all entangled states in "Nonlocality for Two Particles without Inequalities for Almost All Entangled States" [9]. Then we shall present Sheldon Goldstein's formulation of Hardy's argument, originally presented in Goldstein's paper "Nonlocality without inequalities for almost all entangled states for two particles" [10]. In each case, we will use the mathematical framework of quantum mechanics to calculate the probabilities of various measurement outcomes. We will then demonstrate that a local hidden-variables model cannot reproduce these predictions. Furthermore, we will elucidate the significance of the results from the calculations of the probabilities of various measurement outcomes through explicit descriptions. In addition, we will utilise a notation which distinguishes between the physical quantity, the operator which represents the physical quantity, and the result of a measurement of a physical quantity. After the presentation of each argument, its limitations will be considered and comments on its important qualities will be made.

2 Theory

The mathematics and terminology required to follow the arguments in this thesis is presented in this section. This involves the basics of the mathematical formalism of quantum mechanics, including composite systems, a brief introduction to hidden-variables models and the terms *realism*, *locality*, and *local realism*, the properties of the valuation map, and the notation regarding the probability of the value of a measurement outcome.

2.1 The quantum mechanical formalism

Quantum theory, in its conventional formulation, is built on the theory of Hilbert spaces and (linear) operators [11][12]. A complex Hilbert space $\mathcal{H}$ is a vector space over the field $\mathbb{C}$ equipped with a sesquilinear and positive definite map $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ called the inner product. As is customary in physics, we let the inner product be linear in the second argument and antilinear in the first. Then, the inner product satisfies the following properties.

$$\langle u|v\rangle = \overline{\langle v|u\rangle}, \qquad \text{(Conjugate symmetric)}$$

$$\langle u|\alpha v_1 + \beta v_2\rangle = \alpha \langle u|v_1\rangle + \beta \langle u|v_2\rangle \;\; \forall u, v \in \mathcal{H},\; \alpha, \beta \in \mathbb{C}, \qquad \text{(Linear in the second argument)}$$

$$\langle u|u\rangle \geq 0 \quad \forall u \in \mathcal{H},\; \text{and}\; \langle u|u\rangle = 0 \iff u = 0. \qquad \text{(Positive definite)}$$

Following the common bra-ket notation, or Dirac notation, we will often label elements of $\mathcal{H}$ as kets, e.g. $|u\rangle$, and their duals as bras, e.g. $\langle u|$, which is possible due to the Riesz representation theorem. The dual, $\langle u|$, is a linear functional which maps elements of $\mathcal{H}$ to $\mathbb{C}$, $\langle u| : \mathcal{H} \to \mathbb{C}$, by $|v\rangle \mapsto \langle u|v\rangle$. The norm induced by the inner product, $\||u\rangle\| := \sqrt{\langle u|u\rangle}$, defines the length of vectors in the Hilbert space, and an element $|u\rangle \in \mathcal{H}$ such that $\||u\rangle\| = \sqrt{\langle u|u\rangle} = 1$ is called a unit vector. Pure quantum states of a system are unit vectors in the Hilbert space, and the state of a quantum system is denoted $|\psi\rangle$. In this thesis, we will consider only pure quantum states in finite-dimensional Hilbert spaces, hereafter referred to simply as states.

An operator $\widehat{A}$ on the Hilbert space $\mathcal{H}$ is a linear map $\widehat{A} : \mathcal{H} \to \mathcal{H}$. A self-adjoint operator is a linear operator that satisfies $\langle \widehat{A}u|v\rangle = \langle u|\widehat{A}v\rangle$ for all u and v in $\mathcal{H}$, and the set of all self-adjoint operators on a Hilbert space $\mathcal{H}$ is denoted $\mathcal{L}_S(\mathcal{H})$. In addition, any self-adjoint operator has real eigenvalues. The set of eigenvalues of an operator will sometimes be referred to as the spectrum of the operator. In standard quantum mechanics, self-adjoint operators on the Hilbert space are called observables, and measurable physical quantities are represented by observables. Notationally, the physical quantity A is represented by the observable $\widehat{A}$. The result of a measurement of a physical quantity A is in the spectrum of the observable $\widehat{A}$ representing the physical quantity, however, the results of measurements and the probabilities of the different outcomes will be considered in more detail in a following section.

A class of operators called projection operators are self-adjoint operators which satisfy $\widehat{Q} = \widehat{Q}^2$. A rank one projection operator is a linear map $|u\rangle\langle u| : \mathcal{H} \to \mathcal{H}$, which maps elements of $\mathcal{H}$ to itself by $|v\rangle \mapsto \langle u|v\rangle |u\rangle$. By the spectral theorem, any self-adjoint operator with non-degenerate eigenvalues can be written as a sum of projection operators onto its eigenspaces multiplied with the corresponding eigenvalue for that eigenspace, that is, $\widehat{A} = \sum_i a_i \widehat{Q}_i$. The spectral theorem holds for self-adjoint operators with degenerate eigenvalues, but we will limit our discussion to the non-degenerate case.

In quantum mechanics a composite quantum system, that is, a system of more than one particle, is described by the tensor product of the individual systems. The tensor product satisfies

$$\begin{aligned} c\left(|\phi\rangle \otimes |\varphi\rangle\right) &= \left(c\,|\phi\rangle\right) \otimes |\varphi\rangle = |\phi\rangle \otimes \left(c\,|\varphi\rangle\right), \\ |\phi\rangle \otimes \left(|\varphi\rangle + |\psi\rangle\right) &= |\phi\rangle \otimes |\varphi\rangle + |\phi\rangle \otimes |\psi\rangle, \\ \left(|\phi\rangle + |\psi\rangle\right) \otimes |\varphi\rangle &= |\phi\rangle \otimes |\varphi\rangle + |\psi\rangle \otimes |\varphi\rangle, \end{aligned}$$

for all $|\phi\rangle \in \mathcal{H}_1$, $|\varphi\rangle \in \mathcal{H}_2$, and $c \in \mathbb{C}$. The tensor product space, $\mathcal{H}_1 \otimes \mathcal{H}_2$, of the individual vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, is a complex Hilbert space $\mathcal{H}$ with an inner product $\langle \phi_1 \otimes \varphi_1 | \phi_2 \otimes \varphi_2 \rangle \coloneqq \langle \phi_1 | \phi_2 \rangle \langle \varphi_1 | \varphi_2 \rangle$. As an example, consider a composite quantum system described by two complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, with orthonormal bases $\mathcal{B}_1 = \left\{|\phi_i\rangle\right\}$ and $\mathcal{B}_2 = \left\{\left|\varphi_j\right\rangle\right\}$. Then, $\mathcal{H}$ has an orthonormal basis $\mathcal{B} = \left\{|\phi\rangle\,|\varphi\rangle \;\middle|\; \forall\,|\phi\rangle \in \mathcal{B}_1, |\varphi\rangle \in \mathcal{B}_2\right\}$, where $|\phi\rangle\,|\varphi\rangle \coloneqq |\phi\rangle \otimes |\varphi\rangle$. Note that the dimension of the vector space $\mathcal{H}$ is the product of the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$, $\dim(\mathcal{H}) = \dim(\mathcal{H}_1)\dim(\mathcal{H}_2)$. Operators on $\mathcal{H}$ may be defined through $\widehat{A}_1\widehat{B}_2 \coloneqq \widehat{A} \otimes \widehat{B}$, where $\widehat{A}$ and $\widehat{B}$ are operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. An element $|\chi\rangle$ of $\mathcal{H}$ can be expressed as

$$|\chi\rangle = \sum_{i,j} c_{ij}\,|\phi_i\rangle\,\left|\varphi_j\right\rangle.$$

where the elements c_{ij} form a $\dim(\mathcal{H}_1) \times \dim(\mathcal{H}_2)$ matrix C [11]. The matrix C has a singular value decomposition $C = UDV$, where U and V are unitary matrices and D is a diagonal matrix. The diagonal entries in the diagonal matrix D are non-negative real numbers called the singular values of C and are denoted by λ_k. If we assume that $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = d$, then, $|\chi\rangle$ can be written as

$$|\chi\rangle = \sum_{i,j=1}^{d}\left(\sum_{k,l=1}^{d} U_{kl}\lambda_k V_{kl}\delta_{kl}\right)|\phi_i\rangle\,\left|\varphi_j\right\rangle = \sum_{k=1}^{d}\lambda_k\left(\sum_{i=1}^{d} U_{ik}\,|\phi_i\rangle\right)\otimes\left(\sum_{j=1}^{d} V_{kj}\,\left|\varphi_j\right\rangle\right).$$

A different choice of bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{B}_1' = \{u_n\}$ and $\mathcal{B}_2' = \{v_m\}$ such that $u_n \coloneqq \sum_{i=1}^{d} U_{ik}\,|\phi_i\rangle$, and $v_m \coloneqq \sum_{j=1}^{d} V_{kj}\,\left|\varphi_j\right\rangle$, form an orthonormal basis for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then, $\mathcal{B}' = \left\{|u\rangle\,|v\rangle \;\middle|\; \forall\,|u\rangle \in \mathcal{B}_1', |v\rangle \in \mathcal{B}_2'\right\}$ is an orthonormal basis for $\mathcal{H}$. Therefore, the vector $|\chi\rangle$, written in terms of the vectors in the $\mathcal{B}'$ basis is

$$|\chi\rangle = \sum_{k=1}^{m}\lambda_k\,|u_k\rangle\,|v_k\rangle,$$

which is known as the Schmidt decomposition, where m is called its Schmidt rank. The vector $|\chi\rangle$ is called factorised if $m = 1$, that is, if $|\chi\rangle = \lambda_k\,|u\rangle\,|v\rangle$, otherwise it is called entangled. Entangled vectors where the

Schmidt coefficient λ_k is equal for all k are called maximally entangled, and maximally entangled states are maximally entangled unit vectors.

2.2 Local realism and hidden-variables models

Local realism is comprised of two concepts *locality* and *realism.* Realism is the concept of physical objects having definite values for their properties (physical quantities), even when those properties are not being observed or measured, and that those properties are independent of the act of observation. The authors of [1] define a sufficient *criterion of reality* as

> "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.".

This criterion, unlike determinism, doesn't assert that all elements of reality have predictable values. Instead, it contends that if a value is predictable, there must be an element of reality to account for this predictability [13]. (The discussion of determinism is beyond the scope of this thesis, but an introduction to determinism is presented in [14]).

The principle of locality refers to the assumption that any physical effects or interactions between objects occur only through local interactions. As Albert Einstein says in his autobiographical notes [15],

> "But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former.".

That is, any interaction with system S_1, has no effect on system S_2, except through local interactions. The conjunction of locality and realism is referred to by the term *local realism.*

The claim of hidden-variables theories is that different outcomes of measurements obtained from identical quantum states are due to the fact that the system is in a different *microstate*, determined by some unmeasurable quantity or quantities known as hidden variables [16]. A state ϕ in a hidden-variables model is a function of the quantum state ψ and the hidden variables ξ, that is, $\phi(\psi, \xi)$. Furthermore, hidden-variables models, with the assumption of *faithful measurements*, claim that the outcome of a measurement reveals the hidden value of the physical quantity without disturbing it. Moreover, such models, when assuming the principle of locality, posit that the measurement of a physical quantity in one system does not influence the hidden values of physical quantities in other spatially separated systems. In contrast, quantum mechanics predicts the result of a measurement of a physical quantity in a given quantum state as a probability distribution over the spectrum of the observable representing the physical quantity. It also proposes that the values of physical quantities in entangled systems are correlated in a non-local manner. The predictions of quantum mechanics is in agreement with experiments, and therefore, any hidden-variables model which seeks to describe physical reality must reproduce those predictions. However, as will be shown in this thesis, no hidden-variables model which assumes faithful measurements, and whose predictions fulfil the criterion of reality and locality, can reproduce the predictions of quantum mechanics.

2.3 Valuation map

In this section the definition of the valuation map, e.g. [17], will be presented, and some important properties will be derived. In quantum theory, the value of a measurement outcome of a physical quantity A represented by the observable $\widehat{A}$, in a quantum state ψ, is given by a valuation map which maps observables to real-valued scalars, that is, $v_\psi : \mathcal{L}_S(\mathcal{H}) \to \mathbb{R}$, where $\mathcal{L}_S(\mathcal{H})$ is the set of observables on the Hilbert space $\mathcal{H}$. As we will see, if the valuation map is required to satisfy some mild conditions, it cannot assign pre-existing values to physical quantities which satisfy the criterion of reality in a manner which reproduces the predictions of quantum mechanics.

The valuation map is assumed to have the property that the valuation of an observable is in the spectrum of the observable,

$$v_\psi(\widehat{A}) \in \operatorname{spec}(\widehat{A}). \tag{SPEC}$$

In the case of an observables on a finite dimensional Hilbert space, the spectrum of the observable is the set of eigenvalues of the observable. As a second assumption, the valuation map is assumed to satisfy the functional relation

$$v_\psi\left(f_o(\widehat{A})\right) = f\left(v_\psi(\widehat{A})\right), \tag{FUNC}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $f_o(\widehat{A}) := \sum_i f(a_i) Q_i$. Note that for any self-adjoint operator, that is $A = A^*$, and $f : \mathbb{R} \to \mathbb{R}$, $f_o(A) = f_o(A)^*$. As a consequence of (SPEC) and (FUNC) the valuation map satisfies the multiplicative property

$$v_\psi(\widehat{A})\, v_\psi(\widehat{B}) = v_\psi(\widehat{A}\widehat{B}), \tag{MULT}$$

for commuting observables $\widehat{A}$ and $\widehat{B}$, that is $\widehat{A}$ and $\widehat{B}$ such that $[\widehat{A}, \widehat{B}] := \widehat{A}\widehat{B} - \widehat{B}\widehat{A} = 0$. To show that this property is satisfied we will first show that commuting observables share an eigenvector basis. Consider an arbitrary eigenstate $|\psi_i\rangle$ of $\widehat{A}$, then, for commuting observables $\widehat{A}$ and $\widehat{B}$, $\widehat{A}\widehat{B}\,|\psi_i\rangle = \widehat{B}\widehat{A}\,|\psi_i\rangle = a_i\widehat{B}\,|\psi_i\rangle$. Therefore, $\widehat{B}\,|\psi_i\rangle$ is an eigenstate of $\widehat{A}$ with eigenvalue a_i, and since $\widehat{B}\,|\psi_i\rangle$ and $|\psi_i\rangle$ is both eigenstates of $\widehat{A}$, they are related by a constant b_i. Then, $\widehat{B}\,|\psi\rangle = b_i\,|\psi\rangle$, which means that $|\psi_i\rangle$ is an eigenstate of $\widehat{B}$ with eigenvalue b_i. The argument holds for all eigenstates of $\widehat{A}$ and $\widehat{B}$, and therefore, $\widehat{A}$ and $\widehat{B}$ share an eigenvector basis. Since $\widehat{A}$ and $\widehat{B}$ share an eigenvector basis, the spectral decomposition of $\widehat{A}$ and $\widehat{B}$ share a set of projection operators $\{\widehat{Q}_i\}$ such that $\widehat{A} = \sum_i a_i\widehat{Q}_i$ and $\widehat{B} = \sum_i b_i\widehat{Q}_i$. Consider an operator $\widehat{C} = \sum_i c_i\widehat{Q}_i$, and two functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $g(c_i) = a_i$ and $h(c_i) = b_i$. Then, $\widehat{A}$ and $\widehat{B}$ are related to the observable $\widehat{C}$ through $g_o(\widehat{C}) = \sum_i g(c_i)\widehat{Q}_i = \widehat{A}$ and $h_o(\widehat{C}) = \sum_i h(c_i)\widehat{Q}_i = \widehat{B}$. Therefore, $v_\psi(\widehat{A}\widehat{B})$ can be written as $v_\psi\left(g_o(\widehat{C})\, h_o(\widehat{C})\right)$. Consider a function $\Omega_o(\widehat{C}) := \sum_i \Omega(c_i)\widehat{Q}_i$, where $\Omega(c_i) := g(c_i)\, h(c_i)$. Then, $v_\psi\left(g_o(\widehat{C})\, h_o(\widehat{C})\right) = v_\psi\left(\Omega_o(\widehat{C})\right)$, which by (FUNC) is equal to $\Omega\left(v_\psi(\widehat{C})\right)$. The definition of Ω gives $\Omega\left(v_\psi(\widehat{C})\right) = g\left(v_\psi(\widehat{C})\right) h\left(v_\psi(\widehat{C})\right)$, which by (FUNC) is equal to $v_\psi\left(g_o(\widehat{C})\right) v_\psi\left(h_o(\widehat{C})\right)$. Substituting in $g_o(\widehat{C}) = \widehat{A}$ and $h_o(\widehat{C}) = \widehat{B}$, the expression is shown to be equal to $v_\psi(\widehat{A})\, v_\psi(\widehat{B})$. Then it is shown that $v_\psi(\widehat{A}\widehat{B}) = v_\psi(\widehat{A})\, v_\psi(\widehat{B})$.

2.4 Probability notation

In the quantum mechanical formalism, by the Born rule, the probability of obtaining the outcome a_i in a measurement of the physical quantity A represented by the observable $\widehat{A}$ in the quantum state $|\psi\rangle$, that is $v_\psi(\widehat{A}) = a_i$, is

$$P_\psi^A(a_i) = \left\|\widehat{Q}_i\,|\psi\rangle\right\|^2,$$

where $\widehat{Q}_i$ is the projection onto the a_i eigenspace of $\widehat{A}$. Note that, if the quantum state $|\psi\rangle$ is an eigenstate of the observable $\widehat{A}$ such that $\widehat{A}\,|\psi\rangle = a_i\,|\psi\rangle$, then $v_\psi(\widehat{A}) = a_i$ with probability equal to unity, and consequently there exist an element of physical reality corresponding to the physical quantity A with the value a_i. More directly, the criterion of reality is satisfied, and therefore, we say that the physical quantity A has the value a_i in the quantum state $|\psi\rangle$. Similarly for a two particle system, for physical quantities A and B of particle 1 and particle 2 respectively, represented by the operators $\widehat{A} = \sum_i a_i\widehat{Q}_i$ and $\widehat{B} = \sum_j b_j\widehat{R}_j$. The probability of obtaining the outcome a_i and b_i in a joint measurement of A and B in the quantum state $|\psi\rangle$ is

$$P_{\psi}^{A,B}(a_i, b_i) = \left\| \widehat{Q}_{1i}\widehat{R}_{2i} |\psi\rangle \right\|^2,$$

where $\widehat{Q}_{1i}\widehat{R}_{2i} := \widehat{Q}_i \otimes \widehat{R}_i$, with $\widehat{Q}_i$ as the projection onto the a_i eigenspace of $\widehat{A}$ for particle 1 and $\widehat{R}_i$ as the projection onto the b_i eigenspace of $\widehat{B}$ for particle 2. In the following sections, the questions which will be answered concern the probability of different valuations of projections, and therefore we will limit the discussion of probabilities to cases where the observables are projections, in which case $v_\psi(\widehat{Q}) \in \{0,1\}$, for a projection operator $\widehat{Q}$. If the observables on particle 1 and particle 2 are two projection operators $\widehat{Q}$ and $\widehat{R}$ respectively, representing physical quantities Q and R, then the probability of obtaining the outcome 1 for both quantities, that is, $v_\psi(\widehat{Q}) = 1$ and $v_\psi(\widehat{R}) = 1$ may be simplified to $v_\psi(\widehat{Q})\, v_\psi(\widehat{R}) = v_\psi(\widehat{Q}_1\widehat{R}_2) = 1$ by use of the property (MULT). Then, the probability of obtaining the outcome 1 for both Q and R is

$$P_{\psi}^{Q_1R_2}(1) = \left\| \widehat{Q}_1\widehat{R}_2 |\psi\rangle \right\|^2.$$

If $P_{\psi}^{Q_1R_2}(1) = 1$, then the criterion of reality is satisfied, and therefore both Q and R have value 1 in the quantum state $|\psi\rangle$. If $P_{\psi}^{Q_1R_2}(1) = 0$, then $P_{\psi}^{Q_1R_2}(0) = 1$, which means that $v_\psi(Q_1R_2) = v_\psi(Q_1)\, v_\psi(R_2) = 0$, in which case it is shown that Q_1 and R_2 cannot simultaneously have the value 1. The probability of obtaining the outcome 0 for a projection operator $\widehat{Q}$, is equal to the probability of obtaining the outcome 1 for the complement operator $\widehat{Q}^{\complement} := \mathbb{I} - \widehat{Q}$, since there are only two possible outcomes for a valuation of a projection, that is, $v_\psi(\widehat{Q}) \in \{0,1\}$. If $P_{\psi}^{Q^{\complement}}(1) = 1$, then $P_{\psi}^{Q}(1) = 0$ and $P_{\psi}^{Q}(0) = 1$, in which case, the complement of the physical quantity Q satisfies the criterion of reality, and we say that Q has value 0 in the quantum state $|\psi\rangle$. For two projection operators $\widehat{Q}$ and $\widehat{R}$, on particle 1 and particle 2 respectively, the probability of obtaining the outcome 0 for both physical quantities Q and R, that is, $v_\psi(\widehat{Q}) = 0$ and $v_\psi(\widehat{R}) = 0$, is

$$P_{\psi}^{Q_1^{\complement}R_2^{\complement}}(1) = \left\| \widehat{Q}_1^{\complement}\widehat{R}_2^{\complement} |\psi\rangle \right\|^2 = \left\| (\mathbb{I} - \widehat{Q}_1 - \widehat{R}_2 + \widehat{Q}_1\widehat{R}_2) |\psi\rangle \right\|^2.$$

If $P_{\psi}^{Q_1^{\complement}R_2^{\complement}}(1) = 1$, then the physical quantities Q and R both have the value 0 in the quantum state $|\psi\rangle$. It is also possible to define conditional probabilities, that is, the probability of obtaining a valuation $v_\psi(\widehat{Q}) = i$, given that $v_\psi(\widehat{R}) = j$. The definition of conditional probabilities

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

shows that the probability of obtaining $v_\psi(\widehat{Q}) = 1$, given $v_\psi(\widehat{R}) = 1$, for the physical quantities Q and R, is

$$P_{\psi}^{Q|R}(1 \mid 1) = \frac{\left\| \widehat{Q}_1\widehat{R}_2 |\psi\rangle \right\|^2}{\left\| \widehat{R}_2 |\psi\rangle \right\|^2}, \text{ if } \left\| \widehat{R}_2 |\psi\rangle \right\|^2 \neq 0.$$

If $P_{\psi}^{Q|R}(1 \mid 1) = 1$, then the physical quantity Q has value 1 if the physical quantity R has value 1. Note that for probabilities where $v_\psi(\widehat{Q}) = 0$ or $v_\psi(\widehat{R}) = 0$ is considered, a change of operator to $\widehat{Q}^{\complement}$ or $\widehat{R}^{\complement}$, allows the probability to be written in the form given above.

We now turn our attention to the argument put forward by Lucien Hardy [8], his extension to the argument [9], and Goldstein's formulation of Hardy's argument [10].

3 Hardy's argument

In his articles "Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Realistic Theories" [8] and "Nonlocality for Two Particles without Inequalities for Almost All Entangled States" [9], Lucien Hardy presented arguments which show that the predictions of quantum mechanics are inconsistent with the predictions of a class of hidden-variables models. The inconsistency is shown by defining a set of observables that, in a specific state, have no consistent assignment of hidden variables that will reproduce the results calculated from the quantum mechanical formalism. Given that the predictions of the quantum mechanical formalism align with experimental results, any competing theory seeking to describe physical reality must likewise replicate these predictions. Therefore, the class of hidden-variables models shown to be inconsistent with the predictions of quantum mechanics is effectively ruled out.

3.1 The two Mach-Zehnder interferometer argument

In this subsection, a detailed account of Hardy's argument from [8] is presented, and the contradiction which the argument arrives at is shown, using the valuation function. The argument involves two spin-$1/2$ particles, one electron and one positron, each passing through respective Mach-Zehnder interferometers consisting of two beam-splitters (BS). The interferometers are set up such that they overlap, allowing the two particles to annihilate with each other at a point P, as shown in Figure 1. The figure is made to look similar to the figure in [8], but with different notation. To show Hardy's argument, the development of the state $|\psi\rangle$ from its initial state to its final state before detection is derived in four different cases, where the difference between the cases is the placement or removal of beam-splitters in the second layer of the Mach-Zehnder interferometers, that is, the placement or removal of BS_1^2 or BS_2^2. The different configurations lead to different states before detection, and to different valuations of the observables $\widehat{C}_1$, $\widehat{D}_1$, $\widehat{C}_2$, and $\widehat{D}_2$. From the valuations of the observables in the four different cases, one can observe that there is no consistent valuation of the observables in a local hidden variables model that reproduce the predictions of quantum mechanics.

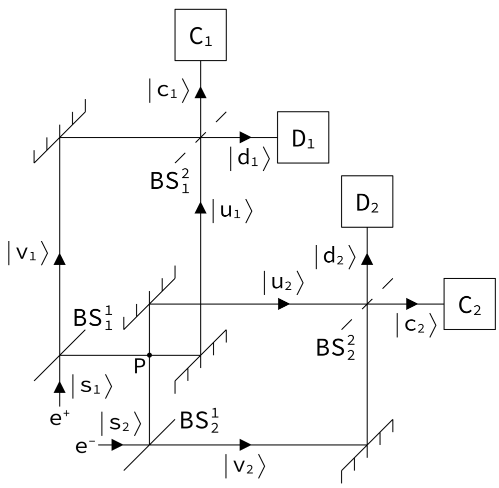

Figure 1: Two overlapping Mach Zehnder interferometers.

3.1.1 Definition of bases, beam-splitters, and initial state

The thought experiment involves two spin-$1/2$ particles – a positron and an electron, which we label as particle 1 and particle 2 respectively. The quantum state of each individual particle is a unit vector in a 2-dimensional complex Hilbert space $\mathcal{H}_n \cong \mathbb{C}^2$, where n refers to the number assigned to the particles, and $n \in \{1, 2\}$. Let $\mathcal{B}_n^1$ denote the orthonormal basis for the initial state of particle n, with $\mathcal{B}_n^1 = \{|r_n\rangle, |s_n\rangle\}$, where $|r_n\rangle = \binom{1}{0}$ and $|s_n\rangle = \binom{0}{1}$ is the standard basis for particle n. The initial state of the positron, particle 1, is $|s_1\rangle$, and the initial state of the electron, particle 2, is $|s_2\rangle$. The state of the composite system, consisting of the two particles, is described by the tensor product of the individual systems. Thus, the state of the composite system is in the 4-dimensional complex Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and the initial state is $|\psi\rangle = |s_1\rangle |s_2\rangle$. After particle n passes through a beam-splitter in the first layer, BS_n^1, we choose to represent the state of the particle as a linear combination of the vectors in the basis $\mathcal{B}_n^2 = \{|v_n\rangle, |u_n\rangle\}$. Further, after particle n passes through the second layer, BS_n^2, we choose to represent the state of the particle is described as a linear combination of the vectors in the basis $\mathcal{B}_n^3 = \{|c_n\rangle, |d_n\rangle\}$. The state of the composite system is described in either of the three bases $\mathcal{B}^1 = \{|r_1\rangle|r_2\rangle, |r_1\rangle|s_2\rangle, |s_1\rangle|r_2\rangle, |s_1\rangle|s_2\rangle\}$, $\mathcal{B}^2 = \{|u_1\rangle|u_2\rangle, |u_1\rangle|v_2\rangle, |v_1\rangle|u_2\rangle, |v_1\rangle|v_2\rangle\}$, or $\mathcal{B}^3 = \{|c_1\rangle|c_2\rangle, |c_1\rangle|d_2\rangle, |d_1\rangle|c_2\rangle, |d_1\rangle|d_2\rangle\}$. The action of a beam-splitter in the first layer of the interferometer is

$$F_n : |s_n\rangle \to \left(i|u_n\rangle + |v_n\rangle\right)/\sqrt{2},$$

and the action of a beam-splitter in the second layer of the interferometer, that is BS_1^2 or BS_2^2, is

$$S_n : \begin{cases} |u_n\rangle \to \left(|c_n\rangle + i|d_n\rangle\right)/\sqrt{2} \\ |v_n\rangle \to \left(i|c_n\rangle + |d_n\rangle\right)/\sqrt{2} \end{cases} .$$

Furthermore, the absence of a beam-splitter in the second layer, that is, the absence of BS_1^2 or BS_2^2, will have the effect

$$X_n : \begin{cases} |u_n\rangle \to |c_n\rangle \\ |v_n\rangle \to |d_n\rangle \end{cases} .$$

Note that n, in the definition of F_n, S_n, and X_n, indicates which particle the transformation acts upon, and consequently which factor of the tensor product the transformation acts upon. Having defined the bases, initial state of the system, and action of the beam-splitters, the state before detection of can be derived.

3.1.2 Transformations of the state

Let the initial state of the system be

$$|\psi\rangle = |s_1\rangle|s_2\rangle .$$

The state of the composite system will be derived in four different cases, where the configurations of the interferometers of the second layer of beam-splitters are different in each case. In all four cases the beam-splitters in the first layer are present, but a beam-splitter in the second layer may be present or absent. The four cases are

1. Neither beam-splitter in the second layer is present.
2. BS_1^2 is present, but BS_2^2 is absent.
3. BS_2^2 is present, but BS_1^2 is absent.
4. Both beam-splitters in the second layer are present.

In each case, let $|\psi_m\rangle$ denote the state of the system before detection, where m indicates which case the state corresponds to. Then, the state before detection in the four different cases are

$$|\psi_1\rangle = (X_1X_2)(F_1F_2)|\psi\rangle, \ |\psi_2\rangle = (S_1X_2)(F_1F_2)|\psi\rangle, \ |\psi_3\rangle = (X_1S_2)(F_1F_2)|\psi\rangle, \ |\psi_4\rangle = (S_1S_2)(F_1F_2)|\psi\rangle .$$

In each case the state of the system after passing through the first layer of the beam-splitters is $F_1F_2|\psi\rangle$. Therefore, to simplify future calculations, the state of the system after passing through the first layer of beam-splitters will be denoted $|\psi'\rangle$, where

$$|\psi'\rangle = F_1 |s_1\rangle F_2 |s_2\rangle = \frac{1}{2}\Big(- |u_1\rangle |u_2\rangle + i |u_1\rangle |v_2\rangle + i |v_1\rangle |u_2\rangle + |v_1\rangle |v_2\rangle \Big).$$

However, the state $|u_1\rangle |u_2\rangle$ expresses a state where the positron and the electron annihilate each other. Therefore, we rewrite $|u_1\rangle |u_2\rangle$ as $|\gamma\rangle$, and get

$$|\psi'\rangle = \frac{1}{2}\Big(- |\gamma\rangle + i |u_1\rangle |v_2\rangle + i |v_1\rangle |u_2\rangle + |v_1\rangle |v_2\rangle \Big).$$

Then, the state before detection in each of the four cases is

$$|\psi_1\rangle = X_1 X_2 |\psi'\rangle, \quad |\psi_2\rangle = S_1 X_2 |\psi'\rangle, \quad |\psi_3\rangle = X_1 S_2 |\psi'\rangle, \quad |\psi_4\rangle = S_1 S_2 |\psi'\rangle.$$

The state before detection in case 1 is

$$|\psi_1\rangle = X_1 X_2 |\psi'\rangle = \frac{1}{2}\Big(- |\gamma\rangle + i |c_1\rangle |d_2\rangle + i |d_1\rangle |c_2\rangle + |d_1\rangle |d_2\rangle \Big). \tag{1}$$

The state before detection in case 2 is

$$|\psi_2\rangle = S_1 X_2 |\psi'\rangle = \frac{1}{2\sqrt{2}}\Big(-\sqrt{2} |\gamma\rangle - |c_1\rangle |c_2\rangle + 2i |c_1\rangle |d_2\rangle + |d_1\rangle |c_2\rangle \Big). \tag{2}$$

The state before detection in case 3 is

$$|\psi_3\rangle = X_1 S_2 |\psi'\rangle = \frac{1}{2\sqrt{2}}\Big(-\sqrt{2} |\gamma\rangle - |c_1\rangle |c_2\rangle + i |c_1\rangle |d_2\rangle + 2i |d_1\rangle |c_2\rangle \Big). \tag{3}$$

Finally, the state before detection in case 4 is

$$|\psi_4\rangle = S_1 S_2 |\psi'\rangle = \frac{1}{4}\Big(- |\gamma\rangle - 3 |c_1\rangle |c_2\rangle + i |c_1\rangle |d_2\rangle + i |d_1\rangle |c_2\rangle - |d_1\rangle |d_2\rangle \Big). \tag{4}$$

3.1.3 Definition of observables and calculation of probabilities

We will show that there is a tension between hidden-variables theories and quantum mechanics by providing a set of observables that have no consistent assignment of hidden-variables that will reproduce the results calculated from the quantum mechanical formalism. Let the observables $\widehat{C}_1$, $\widehat{D}_1$, $\widehat{C}_2$, and $\widehat{D}_2$, be defined by

$$\widehat{C}_1 = |c_1\rangle\langle c_1| \otimes \mathbb{I}, \quad \widehat{D}_1 = |d_1\rangle\langle d_1| \otimes \mathbb{I},$$
$$\widehat{C}_2 = \mathbb{I} \otimes |c_2\rangle\langle c_2|, \quad \widehat{D}_2 = \mathbb{I} \otimes |d_2\rangle\langle d_2|,$$

First, the probability of obtaining the outcome $v_{\psi_1}(\widehat{C}_1) = 1$ and $v_{\psi_1}(\widehat{C}_2) = 1$ in a measurement of the physical quantities C_1 and C_2 in the in the state $|\psi_1\rangle$, that is, the probability of $v_{\psi_1}(\widehat{C}_1\widehat{C}_2) = 1$ is

$$P^{C_1 C_2}_{\psi_1}(1) = \left\| \widehat{C}_1 \widehat{C}_2 |\psi_1\rangle \right\|^2 = \left\| \frac{1}{2}\Big(-\widehat{C}_1\widehat{C}_2 |\gamma\rangle + i\widehat{C}_1\widehat{C}_2 |c_1\rangle |d_2\rangle + i\widehat{C}_1\widehat{C}_2 |d_1\rangle |c_2\rangle + \widehat{C}_1\widehat{C}_2 |d_1\rangle |d_2\rangle \Big) \right\|^2 = 0.$$

This shows that the valuation $v_{\psi_1}(\widehat{C}_1\widehat{C}_2) = 1$ is impossible, that is, C_1 and C_2 cannot simultaneously have the value 1 in the state $|\psi_1\rangle$. Second, the probability of obtaining the outcome $v_{\psi_2}(\widehat{C}_2) = 1$ given that $v_{\psi_2}(\widehat{D}_1) = 1$ in the state $|\psi_2\rangle$ is

$$P^{C_2|D_1}_{\psi_2}(1 \mid 1) = \frac{\left\| \widehat{C}_2 \widehat{D}_1 |\psi_2\rangle \right\|^2}{\left\| \widehat{D}_1 |\psi_2\rangle \right\|^2},$$

where

$$\widehat{D}_1 |\psi_2\rangle = \frac{1}{2\sqrt{2}}\Big(-\sqrt{2}\widehat{D}_1 |\gamma\rangle - \widehat{D}_1 |c_1\rangle |c_2\rangle + 2i\widehat{D}_1 |c_1\rangle |d_2\rangle + \widehat{D}_1 |d_1\rangle |c_2\rangle \Big) = \frac{1}{2\sqrt{2}} |d_1\rangle |c_2\rangle .$$

Then,

$$P_{\psi_2}^{C_2|D_1}(1 \mid 1) = \left\| \frac{2\sqrt{2}}{2\sqrt{2}} \widehat{C}_2 |d_1\rangle |c_2\rangle \right\|^2 = \big\| |d_1\rangle |c_2\rangle \big\|^2 = 1.$$

Therefore, if $v_{\psi_2}(\widehat{D}_1) = 1$ then $v_{\psi_2}(\widehat{C}_2) = 1$ in the state $|\psi_2\rangle$, that is, $v_{\psi_2}(\widehat{D}_1) = 1 \implies v_{\psi_2}(\widehat{C}_2) = 1$. Third, the probability of obtaining the outcome $v_{\psi_3}(\widehat{C}_2) = 1$ given that $v_{\psi_3}(\widehat{D}_2) = 1$ in the state $|\psi_3\rangle$ is

$$P_{\psi_3}^{C_1|D_2}(1 \mid 1) = \frac{\big\| \widehat{C}_1 \widehat{D}_2 |\psi_3\rangle \big\|^2}{\big\| \widehat{D}_2 |\psi_3\rangle \big\|^2},$$

where

$$\widehat{D}_2 |\psi_3\rangle = \frac{1}{2\sqrt{2}}\Big(-\sqrt{2}\widehat{D}_2 |\gamma\rangle - \widehat{D}_2 |c_1\rangle |c_2\rangle + i\widehat{D}_2 |c_1\rangle |d_2\rangle + 2i\widehat{D}_2 |d_1\rangle |c_2\rangle \Big) = \frac{i}{2\sqrt{2}} |c_1\rangle |d_2\rangle .$$

Then,

$$P_{\psi_3}^{C_1|D_2}(1 \mid 1) = \left\| \frac{i2\sqrt{2}}{2\sqrt{2}} \widehat{C}_1 |c_1\rangle |d_2\rangle \right\|^2 = \big\| i |c_1\rangle |d_2\rangle \big\|^2 = 1.$$

Bearing a similarity to the last result, if $v_{\psi_2}(\widehat{D}_2) = 1$ then $v_{\psi_2}(\widehat{C}_1) = 1$ in the state $|\psi_3\rangle$, that is, $v_{\psi_3}(\widehat{D}_2) = 1 \implies v_{\psi_3}(\widehat{C}_1) = 1$. Finally, the probability of obtaining the outcome $v_{\psi_4}(\widehat{D}_1) = 1$ and $v_{\psi_4}(\widehat{D}_2) = 1$ in a measurement of the physical quantities D_1 and D_2 in the in the state $|\psi_4\rangle$, that is, the probability of obtaining the outcome $v_{\psi_4}(\widehat{D}_1 \widehat{D}_2) = 1$ is

$$P_{\psi_4}^{D_1 D_2}(1) = \big\| \widehat{D}_1 \widehat{D}_2 |\psi_4\rangle \big\|^2 = \left\| -\frac{1}{4} |d_1\rangle |d_2\rangle \right\|^2 = \frac{1}{16}.$$

That is, we expect the outcome $v_{\psi_4}(\widehat{D}_1 \widehat{D}_2) = 1$ to be obtained in $1/16$ of the experiments. In summary,

$$P_{\psi_1}^{C_1 C_2}(1) = 0, \tag{5}$$
$$P_{\psi_2}^{C_2|D_1}(1 \mid 1) = 1, \tag{6}$$
$$P_{\psi_3}^{C_1|D_2}(1 \mid 1) = 1, \tag{7}$$
$$P_{\psi_4}^{D_1 D_2}(1) = 1/16. \tag{8}$$

3.1.4 Hardy's argument

We will now argue that hidden-variables model which satisfies the criterion of reality and locality is inconsistent with the predictions, (5)-(8), calculated from the quantum mechanical formalism. Consider an ensemble of systems prepared in an identical quantum state on which we perform measurements. Further, consider an instance in which the outcome $v_{\psi_4}(\widehat{D}_1 \widehat{D}_2) = 1$ is obtained. Then, by use of the property (MULT), the valuation $v_{\psi_4}(\widehat{D}_1 \widehat{D}_2) = 1$ can be written as $v_{\psi_4}(\widehat{D}_1) \, v_{\psi_4}(\widehat{D}_2) = 1$. The observables $\widehat{D}_1$ and $\widehat{D}_2$ are projections and therefore, for the product $v_{\psi_4}(\widehat{D}_1) \, v_{\psi_4}(\widehat{D}_2)$ to be equal to 1, (SPEC) implies that

$$v_{\psi_4}(\widehat{D}_1) = 1, \text{ and } v_{\psi_4}(\widehat{D}_2) = 1.$$

From the hidden-variables theory perspective, if we assume *faithful measurements*, that is, that we obtain the hidden pre-existing value of the physical quantity which we measure without disturbing it. Then we can say that D_1 and D_2 both have value 1 in the state $|\psi_4\rangle$ in the particular instance which we are considering.

Furthermore, if we assume that the principle of locality holds, then the valuations $v_{\psi_4}(\widehat{D}_1)$ and $v_{\psi_2}(\widehat{D}_1)$ should have the same value due to the fact that nothing that is local to particle 1 has changed between cases 2 and 4. That is, in both cases, particle 1 interacts with the beam-splitter BS_1^1, and subsequently, it can travel freely through the path labelled $|v_1\rangle$ to BS_1^2, or it can travel through the path labelled $|u_1\rangle$, where it will either annihilate with particle 2 or travel freely. In either case, the only difference between cases 2 and 4 are the placement or absence of BS_2^2, which particle 1 does not interact with. Similarly, $v_{\psi_4}(\widehat{D}_2)$ must be equal to $v_{\psi_3}(\widehat{D}_2)$. That is,

$$v_{\psi_4}(\widehat{D}_1) = 1 \Leftrightarrow v_{\psi_2}(\widehat{D}_1) = 1, \text{ and}$$
$$v_{\psi_4}(\widehat{D}_2) = 1 \Leftrightarrow v_{\psi_3}(\widehat{D}_2) = 1.$$

Then, from (6) and (7),

$$v_{\psi_2}(\widehat{D}_1) = 1 \Leftrightarrow v_{\psi_2}(\widehat{C}_2) = 1, \text{ and}$$
$$v_{\psi_3}(\widehat{D}_2) = 1 \Leftrightarrow v_{\psi_3}(\widehat{C}_1) = 1.$$

However, nothing that is local to particle 2 has changed between cases 2 and 1. Therefore, $v_{\psi_2}(\widehat{C}_2) = v_{\psi_1}(\widehat{C}_2) = 1$. And by similar argument for particle 1 in cases 3 and 1, $v_{\psi_3}(\widehat{C}_1) = v_{\psi_1}(\widehat{C}_1) = 1$. Therefore, $v_{\psi_1}(\widehat{C}_1) = 1$ and $v_{\psi_1}(\widehat{C}_2) = 1$, which by (MULT) implies that

$$v_{\psi_1}(\widehat{C}_1\widehat{C}_2) = 1. \tag{9}$$

Then, there is contradiction between (5) and (9). From the perspective of a hidden-variables theory, in the instances where we find that D_1 and D_2 have value 1, (6) and (7) satisfy the criterion of reality and imply that C_1 and C_2 must simultaneously have value 1, if we assume that the principle of locality holds. However, the complement of (5) satisfies the criterion of reality, and therefore either or both of C_1 and C_2 must have a value of 0, which shows the contradiction.

The contradiction shows that, in $^1/_{16}$'th of the experiments, there is no consistent way of assigning hidden variables to the observables that will reproduce the predictions quantum mechanics. A notable feature of Hardy's argument is that it does not rely on inequalities, as opposed to Bell's inequality [2], but bares similarity to a GHZ-argument [7] in that if the result $v_{\psi_4}(\widehat{D}_1\widehat{D}_2) = 1$ is obtained, then the inconsistency argument follows. However, Hardy's argument is not as strong as the GHZ-argument as it only shows an inconsistency for $^1/_{16}$'th of experiments compared to the GHZ-argument which shows the inconsistency for every run of the experiment.

3.2 Hardy's extension of the argument

In 1993 Hardy published an extension of his argument in the article "Nonlocality for Two Particles without Inequalities for Almost All Entangled States" [9]. In the article he extends the proof from the specific entangled state covered by the Mach-Zehnder interferometer argument, to almost all entangled states. Surprisingly, he discovers that the argument fails if the system is in maximally entangled states, which is curious, as one expects maximally entangled states to exhibit the most non-classical characteristics. Hardy makes his argument by considering an arbitrary initial entangled state in three different bases, from which he presents four different representations of the state, where the four different observables $\widehat{U}_1$, $\widehat{U}_2$, $\widehat{D}_1$, and $\widehat{D}_2$ can be seen to have no consistent hidden-variables assignment. In this section, we will present a detailed derivation of Hardy's extension to the argument, along with comments on why this argument fails in maximally entangled states. The notation will be consistent with the notation employed in the Mach-Zehnder argument, but note that the relation between the bases has changed and that the initial state is an arbitrary entangled state.

3.2.1 Bases and representations of the state

The argument involves two spin-1/2 particles, where the state of each particle is a unit vector in a 2-dimensional complex Hilbert space $\mathcal{H}_n$. The three bases for each particle are $\mathcal{B}_n^1 = \{|r_n\rangle, |s_n\rangle\}$, where $|r_n\rangle = \binom{1}{0}$ and $|s_n\rangle = \binom{0}{1}$ is the standard basis for particle n, $\mathcal{B}_n^2 = \{|u_n\rangle, |v_n\rangle\}$, and $\mathcal{B}_n^3 = \{|c_n\rangle, |d_n\rangle\}$. The state of the composite system is in a 4-dimensional complex Hilbert space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and will described by a linear combination of vectors from the three different bases $\mathcal{B}^1 = \{|r_1\rangle|r_2\rangle, |r_1\rangle|s_2\rangle, |s_1\rangle|r_2\rangle, |s_1\rangle|s_2\rangle\}$, $\mathcal{B}^2 = \{|u_1\rangle|u_2\rangle, |u_1\rangle|v_2\rangle, |v_1\rangle|u_2\rangle, |v_1\rangle|v_2\rangle\}$, or $\mathcal{B}^3 = \{|c_1\rangle|c_2\rangle, |c_1\rangle|d_2\rangle, |d_1\rangle|c_2\rangle, |d_1\rangle|d_2\rangle\}$. The basis vectors in $\mathcal{B}^1$ expressed in terms of the basis vectors in $\mathcal{B}^2$ are

$$|r_n\rangle = b\,|u_n\rangle + i\bar{a}\,|v_n\rangle\,, \quad |s_n\rangle = ia\,|u_n\rangle + \bar{b}\,|v_n\rangle\,, \text{ where } a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1.$$

The basis vectors in $\mathcal{B}^2$ expressed in terms of the basis vectors in $\mathcal{B}^1$ are

$$|u_n\rangle = \bar{b}\,|r_n\rangle - i\bar{a}\,|s_n\rangle\,, \quad |v_n\rangle = -ia\,|r_n\rangle + b\,|s_n\rangle\,,$$

Similarly, the basis vectors in $\mathcal{B}^2$ expressed in terms of the basis vectors in $\mathcal{B}^3$ are

$$|u_n\rangle = A\,|c_n\rangle - B\,|d_n\rangle\,, \quad |v_n\rangle = B\,|c_n\rangle + A\,|d_n\rangle\,,$$

and the basis vectors in $\mathcal{B}^3$ expressed in terms of the basis vectors in $\mathcal{B}^2$ are

$$|c_n\rangle = A\,|u_n\rangle + B\,|v_n\rangle\,, \quad |d_n\rangle = -B\,|u_n\rangle + A\,|v_n\rangle\,,$$

where

$$A = \sqrt{\frac{\alpha\beta}{1 - |\alpha\beta|}}, \text{ and } \quad B = \frac{|\alpha| - |\beta|}{\sqrt{1 - |\alpha\beta|}}, \text{ with } \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 = 1.$$

The state of the composite system in terms of the vectors in the $\mathcal{B}^1$ basis is

$$|\psi\rangle = \alpha\,|r_1\rangle|r_2\rangle - \beta\,|s_1\rangle|s_2\rangle\,.$$

Then, the state $|\psi\rangle$ expressed as a linear combination of the vectors in the $\mathcal{B}^2$ basis is

$$\begin{aligned}|\psi\rangle &= \alpha\left(b\,|u_1\rangle + i\bar{a}\,|v_1\rangle\right)\left(b\,|u_2\rangle + i\bar{a}\,|v_2\rangle\right) - \beta\left(ia\,|u_1\rangle + \bar{b}\,|v_1\rangle\right)\left(ia\,|u_1\rangle + \bar{b}\,|v_1\rangle\right), \\ &= \left(\alpha b^2 + \beta a^2\right)|u_1\rangle|u_2\rangle + i\left(\alpha\bar{a}b - \beta a\bar{b}\right)|u_1\rangle|v_2\rangle + i\left(\alpha\bar{a}b - \beta a\bar{b}\right)|v_1\rangle|u_2\rangle - \left(\alpha\bar{a}^2 + \beta\bar{b}^2\right)|v_1\rangle|v_2\rangle\,.\end{aligned}$$

As a part of Hardy's argument $|\psi\rangle$ should not contain a $|u_1\rangle|u_2\rangle$ term, giving us the condition $\alpha b^2 + \beta a^2 = 0$. By choosing appropriate phases for a and b, the constant $K_1 = 1/\sqrt{|\alpha| + |\beta|}$ can be made to be real. Then, the condition on a and b is

$$a = K_1\sqrt{\alpha}, \text{ and } b = iK_1\sqrt{\beta}.$$

By substituting in for a and b in the expression for $|\psi\rangle$, the expression is rewritten as

$$|\psi\rangle = -\left(\sqrt{\alpha\beta}\,|u_1\rangle|v_2\rangle + \sqrt{\alpha\beta}\,|v_1\rangle|u_2\rangle + \left(|\alpha| - |\beta|\right)|v_1\rangle|v_2\rangle\right),$$

where the factor of -1 can be dropped without affecting the probabilities. Additionally, we observe that the state can be written in the equivalent form

$$|\psi\rangle = \frac{\alpha\beta}{|\alpha| - |\beta|}\left(\left(|u_1\rangle + \frac{|\alpha| - |\beta|}{\sqrt{\alpha\beta}}|v_1\rangle\right)\left(|u_2\rangle + \frac{|\alpha| - |\beta|}{\sqrt{\alpha\beta}}|v_2\rangle\right) - |u_1\rangle|u_2\rangle\right).$$

If we define

$$N = \frac{1 - |\alpha\beta|}{|\alpha| - |\beta|},$$

and note that

$$NA^2 = \frac{\left(1 - |\alpha\beta|\right)(\alpha\beta)}{\left(|\alpha| - |\beta|\right)\left(1 - |\alpha\beta|\right)} = \frac{\alpha\beta}{|\alpha| - |\beta|}, \text{ and } \frac{B}{A} = \frac{\left(|\alpha| - |\beta|\right)\sqrt{1 - |\alpha\beta|}}{\sqrt{\alpha\beta}\sqrt{1 - |\alpha\beta|}} = \frac{|\alpha| - |\beta|}{\sqrt{\alpha\beta}}.$$

Then, the state $|\psi\rangle$ may be expressed by

$$\begin{aligned} |\psi\rangle &= NA^2\left(\left(|u_1\rangle + \frac{B}{A}|v_1\rangle\right)\left(|u_2\rangle + \frac{B}{A}|v_2\rangle\right) - |u_1\rangle|u_2\rangle\right), \\ &= N\Big(\left(A|u_1\rangle + B|v_1\rangle\right)\left(A|u_2\rangle + B|v_2\rangle\right) - A^2|u_1\rangle|u_2\rangle\Big). \end{aligned} \tag{10}$$

By substituting $|c_n\rangle = A|u_n\rangle + B|v_n\rangle$ the state can be rewritten as

$$|\psi\rangle = N\Big(|c_1\rangle|c_2\rangle - A^2|u_1\rangle|u_2\rangle\Big). \tag{11}$$

From the representations (10) and (11) of the state $|\psi\rangle$ it is possible to construct four equivalent representations of the state. The first is achieved by expanding the terms in (10),

$$|\psi\rangle_1 = N\Big(AB|u_1\rangle|v_2\rangle + AB|v_1\rangle|u_2\rangle + B^2|v_1\rangle|v_2\rangle\Big). \tag{12}$$

The second representation is achieved by substituting $|c_2\rangle = A|u_2\rangle + B|v_2\rangle$, and $|u_1\rangle = A|c_1\rangle - B|d_1\rangle$, in (11), giving

$$|\psi\rangle_2 = N\Big(|c_1\rangle\left(A|u_2\rangle + B|v_2\rangle\right) - A^2\left(A|c_1\rangle - B|d_1\rangle\right)|u_2\rangle\Big). \tag{13}$$

The third representation is achieved by substituting $|c_1\rangle = A|u_1\rangle + B|v_1\rangle$, and $|u_2\rangle = A|c_2\rangle - B|d_2\rangle$, in (11), giving

$$|\psi\rangle_3 = N\Big(\left(A|u_1\rangle + B|v_1\rangle\right)|c_2\rangle - A^2|u_1\rangle\left(A|c_2\rangle - B|d_2\rangle\right)\Big). \tag{14}$$

The fourth, and last, representation of the state is achieved by substituting $|u_1\rangle = A|c_1\rangle - B|d_1\rangle$ and $|u_2\rangle = A|c_2\rangle - B|d_2\rangle$ in (11), giving

$$|\psi\rangle_4 = N\Big(|c_1\rangle|c_2\rangle - A^2\left(A|c_1\rangle - B|d_1\rangle\right)\left(A|c_2\rangle - B|d_2\rangle\right)\Big). \tag{15}$$

Given the four different but equivalent representations (12), (13), (14), and (15), of the state $|\psi\rangle$, it is possible to define a set of observables which can be seen to have no possible consistent assignment of hidden variables.

3.2.2 Observables and probabilities

Let the operators representing the four different physical quantities U_1, U_2, D_1, and D_2, be defined as

$$\widehat{U}_1 = |u_1\rangle\langle u_1| \otimes \mathbb{I}, \quad \widehat{D}_1 = |d_1\rangle\langle d_1| \otimes \mathbb{I},$$
$$\widehat{U}_2 = \mathbb{I} \otimes |u_2\rangle\langle u_2|\,, \quad \widehat{D}_2 = \mathbb{I} \otimes |d_2\rangle\langle d_2|\,,$$

The probability of obtaining the outcome $v_\psi(\widehat{U}_1) = 1$ and $v_\psi(\widehat{U}_2) = 1$ is

$$P_\psi^{U_1U_2}(1) = \left\|\widehat{U}_1\widehat{U}_2\,|\psi\rangle_1\right\|^2 = 0.$$

Therefore, U_1 and U_2 cannot simultaneously have value 1. The probability of obtaining the outcome $v_\psi(\widehat{U}_2) = 1$ given that $v_\psi(\widehat{D}_1) = 1$, in the state $|\psi\rangle$ is

$$P_\psi^{U_2|D_1}(1 \mid 1) = \frac{\left\|\widehat{D}_1\widehat{U}_2\,|\psi\rangle_2\right\|^2}{\left\|\widehat{D}_1\,|\psi\rangle_2\right\|^2},$$

where

$$\widehat{D}_1\,|\psi\rangle_2 = N\Big(\widehat{D}_1\,|c_1\rangle\left(A\,|u_2\rangle + B\,|v_2\rangle\right) - A^2\left(A\widehat{D}_1\,|c_1\rangle - B\widehat{D}_1\,|d_1\rangle\right)|u_2\rangle\Big) = NA^2B\,|d_1\rangle\,|u_2\rangle\,.$$

Therefore,

$$P_\psi^{U_2|D_1}(1 \mid 1) = \left\|\frac{NA^2B\widehat{U}_2\,|d_1\rangle\,|u_2\rangle}{NA^2B}\right\|^2 = \left\||d_1\rangle\,|u_2\rangle\right\|^2 = 1.$$

Therefore, if $v_\psi(\widehat{D}_1) = 1$ then $v_\psi(\widehat{U}_2) = 1$ in the state $|\psi\rangle$, that is, $v_\psi(\widehat{D}_1) = 1 \implies v_\psi(\widehat{U}_2) = 1$. Similarly, the probability of obtaining the outcome $v_\psi(\widehat{U}_1) = 1$ given that $v_\psi(\widehat{D}_2) = 1$, in the state $|\psi\rangle$ is

$$P_\psi^{U_1|D_2}(1 \mid 1) = \frac{\left\|\widehat{U}_1\widehat{D}_2\,|\psi\rangle_3\right\|^2}{\left\|\widehat{D}_2\,|\psi\rangle_3\right\|^2},$$

where

$$\widehat{D}_2\,|\psi\rangle_3 = N\Big(\left(A\,|u_1\rangle + B\,|v_1\rangle\right)\widehat{D}_2\,|c_2\rangle - A^2\,|u_1\rangle\left(A\widehat{D}_2\,|c_2\rangle - B\widehat{D}_2\,|d_2\rangle\right)\Big) = NA^2B\,|u_1\rangle\,|d_2\rangle\,.$$

Therefore,

$$P_\psi^{U_1|D_2}(1 \mid 1) = \left\|\frac{NA^2B\widehat{U}_1\,|u_1\rangle\,|d_2\rangle}{NA^2B}\right\|^2 = \left\||u_1\rangle\,|d_2\rangle\right\|^2 = 1.$$

Therefore, if $v_\psi(\widehat{D}_2) = 1$ then $v_\psi(\widehat{U}_1) = 1$ in the state $|\psi\rangle$, that is, $v_\psi(\widehat{D}_2) = 1 \implies v_\psi(\widehat{U}_1) = 1$. The probability of obtaining the outcome $v_\psi(\widehat{D}_1) = 1$ and $v_\psi(\widehat{D}_2) = 1$ is

$$P_\psi^{D_1D_2}(1) = \left\|\widehat{D}_1\widehat{D}_2\,|\psi\rangle_4\right\|^2 = \left|NA^2B^2\right|^2.$$

Note that $\left|NA^2B^2\right|^2 = \left(\frac{|\alpha\beta|\left(|\alpha|-|\beta|\right)}{1-|\alpha\beta|}\right)^2 > 0$ is greater than 0 except when $|\alpha| = |\beta|$, in which case the state is a maximally entangled state. In summary,

$$P_\psi^{U_1U_2}(1) = 0, \tag{16}$$

$$P_\psi^{U_2|D_1}(1 \mid 1) = 1, \tag{17}$$

$$P_\psi^{U_1|D_2}(1 \mid 1) = 1, \tag{18}$$

$$P_\psi^{D_1D_2}(1) = \left|NA^2B^2\right|^2. \tag{19}$$

3.2.3 Hardy's argument

The argument follows the same structure as the argument presented in the two Mach-Zehnder interferometers argument. However, in the argument presented in this section, the four different representations of the state are all equivalent descriptions of one state. Consequently, there is no need to consider the valuations of different states, which makes the argument more immediate. Consider an ensemble of systems, prepared in identical quantum states, on which we perform measurements. The result $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$ will be obtained with probability $\left|NA^2B^2\right|^2$. In experiments where $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$ is obtained, by (MULT) and (SPEC)

$$v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = v_\psi\left(\widehat{D}_1\right)v_\psi\left(\widehat{D}_2\right) = 1 \implies v_\psi\left(\widehat{D}_1\right) = 1, \text{ and } v_\psi\left(\widehat{D}_2\right) = 1.$$

Then, from (17) and (18)

$$v_\psi\left(\widehat{D}_1\right) = 1 \implies v_\psi\left(\widehat{U}_2\right) = 1, \text{ and } v_\psi\left(\widehat{D}_2\right) = 1 \implies v_\psi\left(\widehat{U}_1\right) = 1.$$

Therefore, $v_\psi\left(\widehat{U}_2\right) = 1$ and $v_\psi\left(\widehat{U}_1\right) = 1$, which by (MULT) is equal to $v_\psi\left(\widehat{U}_1\widehat{U}_2\right) = 1$. Therefore,

$$v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1 \implies v_\psi\left(\widehat{U}_1\widehat{U}_2\right) = 1,$$

which contradicts (16).

In the perspective of hidden-variables theories, if we assume *faithful measurements*, and that the principle of locality holds, that is, that a measurement on one particle does not affect the hidden value of the physical quantities of the other particle. Then, in the instances where the outcome $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$ is obtained, we can say that D_1 and D_2 have the hidden value 1, and that (17) and (18) satisfies the criterion of reality which implies that U_1 and U_2 have value 1. However, the complement of (16) satisfies the criterion of reality and imply that either or both of U_1 and U_2 must have a value of 0, which shows the contradiction.

The contradiction shows that there are no hidden-variables assignment which is able to reproduce the predictions of quantum mechanics. Notably, the argument holds for all states of the form $|\psi\rangle = \alpha|r_1\rangle|r_2\rangle - \beta|s_1\rangle|s_2\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$, except when $|\alpha| = |\beta|$, the reason for which will be explained shortly. Therefore, the argument shows that there is a Hardy-type contradiction for almost all entangled states.

3.2.4 The maximum probability of obtaining the contradiction

The maximum probability of obtaining $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$, is found by maximising

$$\left|NA^2B^2\right|^2 = \left(\frac{|\alpha\beta|\left(|\alpha| - |\beta|\right)}{1 - |\alpha\beta|}\right)^2,$$

Recall that α and β are real numbers such that $|\alpha|^2 + |\beta|^2 = 1$. Then, $\left|NA^2B^2\right|^2$ may be written as a function of $x = |\alpha|$ and $y = |\beta|$

$$\left|NA^2B^2\right|^2 = \left(\frac{xy\left(x-y\right)}{1-xy}\right)^2,$$

where x and y are real numbers in the interval $[0,1]$ such that $x^2+y^2=1$. By using the constraint $y=\sqrt{1-x^2}$, $\left|NA^2B^2\right|^2$ can be written as a function

$$f\left(x\right) = \left(\frac{x\sqrt{1-x^2}\left(x-\sqrt{1-x^2}\right)}{1-x\sqrt{1-x^2}}\right)^2.$$

The stationary points are found by solving for x such that

$$\frac{\mathrm{d}}{\mathrm{d}x}f\left(x\right) = \frac{2x\left(x-\sqrt{1-x^2}\right)\left(x^5+x^4\sqrt{1-x^2}-4x^3+2x^2\sqrt{1-x^2}+2x-\sqrt{1-x^2}\right)}{\left(1-x\sqrt{1-x^2}\right)^3} = 0.$$

The solution which give the maximum value of $\left|NA^2B^2\right|^2$ is $x_{max} = \sqrt{\frac{1}{2}\left(1+\sqrt{6\sqrt{5}-13}\right)}$, and therefore the maximum probability of obtaining $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$ is

$$f\left(x_{max}\right) = \frac{1}{2}\left(5\sqrt{5}-11\right) \approx 0.09.$$

Then,

$$|\alpha| = \sqrt{\frac{1}{2}\left(1+\sqrt{6\sqrt{5}-13}\right)} \approx 0.9070, \quad |\beta| = 1-\frac{1}{2}\left(1+\sqrt{6\sqrt{5}-13}\right) \approx 0.4211,$$

and the state in which the probability of obtaining $v_\psi\left(\widehat{D}_1\widehat{D}_2\right) = 1$ is at its maximum is

$$\left|\psi_{max}\right\rangle = 0.9070\left|r_1\right\rangle\left|r_2\right\rangle - 0.4211\left|s_1\right\rangle\left|s_2\right\rangle.$$

3.2.5 Considerations for maximally entangled states

Hardy's argument can be seen to fail in maximally entangled states by considering the representations of the state

$$\begin{aligned}
\left|\psi\right\rangle_1 &= NAB\left|u_1\right\rangle\left|v_2\right\rangle + NAB\left|v_1\right\rangle\left|u_2\right\rangle + NB^2\left|v_1\right\rangle\left|v_2\right\rangle, \\
\left|\psi\right\rangle_2 &= NA\left|c_1\right\rangle\left|u_2\right\rangle + NB\left|c_1\right\rangle\left|v_2\right\rangle - NA^2A\left|c_1\right\rangle\left|u_2\right\rangle - NA^2B\left|d_1\right\rangle\left|u_2\right\rangle, \\
\left|\psi\right\rangle_3 &= NA\left|u_1\right\rangle\left|c_2\right\rangle + NB\left|v_1\right\rangle\left|c_2\right\rangle - NA^2A\left|u_1\right\rangle\left|c_2\right\rangle - NA^2B\left|u_1\right\rangle\left|d_2\right\rangle, \\
\left|\psi\right\rangle_4 &= N\left|c_1\right\rangle\left|c_2\right\rangle - NA^4\left|c_1\right\rangle\left|c_2\right\rangle + NA^3B\left|c_1\right\rangle\left|d_2\right\rangle + NA^3B\left|d_1\right\rangle\left|c_2\right\rangle - NA^2B^2\left|d_1\right\rangle\left|d_2\right\rangle.
\end{aligned}$$

The factors that appear in the expansions of the different representations of the state are NAB, NB^2, NA, NB, NA^3, NA^2B, N, NA^4, NA^3B, and NA^2B^2. Recall that

$$N = \frac{1-|\alpha\beta|}{|\alpha|-|\beta|}, \quad A = \sqrt{\frac{\alpha\beta}{1-|\alpha\beta|}}, \quad B = \frac{|\alpha|-|\beta|}{\sqrt{1-|\alpha\beta|}}.$$

Then it is clear that some of the terms diverge when the state is maximally entangled since $|\alpha| = |\beta|$ in such states. The terms that diverge are the terms which have a factor of either of the following,

$$N = \frac{1-|\alpha\beta|}{|\alpha|-|\beta|}, \quad NA = \frac{\sqrt{\alpha\beta\,(1-|\alpha\beta|)}}{|\alpha|-|\beta|}, \quad NA^3 = \frac{\sqrt{|\alpha\beta|^3}}{\left(|\alpha-|\beta||\right)(1-|\alpha\beta|)}, \quad NA^4 = \frac{|\alpha\beta|^2}{\left(|\alpha|-|\beta|\right)(1-|\alpha\beta|)}.$$

Since some terms diverge, Hardy's argument will not hold in maximally entangled states.

Additionally, the argument fails because the observables commute in general when $|\alpha| = |\beta|$, in which case classical behaviour is expected. In order to show that the observables commute, we express the observables in terms of the vectors in the $\mathcal{B}^1$ basis, and show that the condition $|\alpha| = |\beta|$ gives a commutator equal to zero. The $|u_n\rangle$ vectors expressed in terms of the vectors in the $\mathcal{B}^1$ basis are

$$|u_n\rangle = \bar{b}\,|r_n\rangle - i\bar{a}\,|s_n\rangle\,.$$

Recall that $a = \bar{a} = K_1\sqrt{\alpha}$ and $b = iK_1\sqrt{\beta}$, where $K_1 = 1/\sqrt{|\alpha|+|\beta|}$. Then,

$$|u_n\rangle = -iK_1\left(\sqrt{\beta}\,|r_n\rangle + \sqrt{\alpha}\,|s_n\rangle\right).$$

Therefore, the observables $\widehat{U}_n$ expressed in terms of the vectors in the $\mathcal{B}^1$ basis are

$$\widehat{U}_n = |u_n\rangle\langle u_n| = K_1^2\left(\beta\,|r_n\rangle\langle r_n| + \sqrt{\alpha\beta}\,|r_n\rangle\langle s_n| + \sqrt{\alpha\beta}\,|s_n\rangle\langle r_n| + \alpha\,|s_n\rangle\langle s_n|\right).$$

Similarly, the observables $\widehat{D}_n$ expressed in terms of the vectors in the $\mathcal{B}^1$ basis are

$$|d_n\rangle = -\left(B\bar{b} + Aia\right)|r_n\rangle + (Bia + Ab)\,|s_n\rangle = -i\left(\frac{\sqrt{\beta^3}\,|r_n\rangle - \sqrt{\alpha^3}\,|s_n\rangle}{\sqrt{(1-|\alpha\beta|)\left(|\alpha|+|\beta|\right)}}\right).$$

By simplifying the denominator,

$$\sqrt{(1-|\alpha\beta|)\left(|\alpha|+|\beta|\right)} = |\alpha| - |\alpha|^2|\beta| + |\beta| - |\alpha||\beta|^2 = -\left(|\alpha|^3 + |\beta|^3\right),$$

the expression for $|d_n\rangle$ becomes

$$|d_n\rangle = i\left(\frac{\sqrt{\beta^3}\,|r_n\rangle - \sqrt{\alpha^3}\,|s_n\rangle}{|\alpha|^3 + |\beta|^3}\right) = iK_2\left(\sqrt{\beta^3}\,|r_n\rangle - \sqrt{\alpha^3}\,|s_n\rangle\right),$$

where $K_2 = 1/\left(|\alpha|^3 + |\beta|^3\right)$. Then, the observables $\widehat{D}_n$ expressed in terms of the vectors in the $\mathcal{B}^1$ basis are

$$\widehat{D}_n = |d_n\rangle\langle d_n| = K_2^2\left(\beta^3\,|r_n\rangle\langle r_n| - \sqrt{\alpha^3\beta^3}\,|r_n\rangle\langle s_n| - \sqrt{\alpha^3\beta^3}\,|s_n\rangle\langle r_n| + \alpha^3\,|s_n\rangle\langle s_n|\right).$$

Then, the commutator

$$\frac{[\widehat{U}_n, \widehat{D}_n]}{K_1^2K_2^2} = \left(\sqrt{\alpha^7\beta} - \sqrt{\alpha^3\beta^5} - \sqrt{\alpha\beta^7} + \sqrt{\alpha^5\beta^3}\right)|r_n\rangle\langle s_n| + \left(\sqrt{\alpha^7\beta} - \sqrt{\alpha^5\beta^3} - \sqrt{\alpha^7\beta} + \sqrt{\alpha^3\beta^5}\right)|s_n\rangle\langle r_n|\,.$$

Therefore, when $|\alpha| = |\beta|$, as is the case in maximally entangled states, the commutator $[\widehat{U}_n, \widehat{D}_n] = 0$, and the observables commute. In such cases, classical behaviour is expected when the observables commute. Consequently, an assignment of hidden-variables should be possible, which means that Hardy's argument will not go through. Although this shows that Hardy's method of showing a contradiction does not hold in maximally entangled states, it does not determine whether a an argument may be found by a different method. However, in his thesis "The Mathematical Structure of Non-locality & Contextuality" [18], Shane Mansfield shows that a Hardy-type argument cannot be made in maximally entangled states using projective measurements.

4 Goldstein's formulation of Hardy's argument

In 1994, a streamlined version of the argument made by Hardy in his 1993 article was presented by Sheldon Goldstein in the article "Nonlocality without inequalities for almost all entangled states for two particles" [10]. The argument was streamlined by assuming the state to be of a general form of a desired state of a composite system, specifically a state with no $|u_1\rangle|u_2\rangle$ term, rather than defining relations between a set of bases which describes the same state, as Hardy did in his 1993 article. By considering a general state with no $|u_1\rangle|u_2\rangle$ term, Goldstein avoids making the same constraints as Hardy made in his argument, which generalises the argument. As Goldstein himself puts it: "While Hardy's analysis concerns four observables the choice of each of which is very much constrained by the quantum state, for our argument the choice of one of the observables is almost arbitrary.". In addition to streamlining the proof, Goldstein addresses the question of why the argument fails in maximally entangled states in detail. Goldstein's argument, while concise, employs notation that may lack clarity in differentiating between the observables, the valuation of the observables, and the probability of obtaining a specific value of an observable. Therefore, in the following detailed derivation of Goldstein's version of Hardy's argument, we will utilise the notation used in previous sections to clarify the points mentioned. Additionally, Goldstein's method of showing that the argument holds for all entangled states except maximally entangled states can be challenging to follow. We will substantiate the assertion that the argument remains valid in all entangled states, except maximally entangled ones, while attempting to make it more accessible.

4.1 State, observables, and probabilities

The state under consideration is

$$|\psi\rangle = a\,|v_1\rangle\,|v_2\rangle + b\,|u_1\rangle\,|v_2\rangle + c\,|v_1\rangle\,|u_2\rangle\,, \quad \text{where } abc \neq 0, \text{ and } |a|^2 + |b|^2 + |c|^2 = 1. \tag{20}$$

Let two vectors $|w_1\rangle$ and $|w_2\rangle$ be defined as

$$|w_1\rangle = k_1\Big(a\,|v_1\rangle + b\,|u_1\rangle\Big), \text{ and } |w_2\rangle = k_2\Big(a\,|v_2\rangle + c\,|u_2\rangle\Big),$$

where

$$k_1 = \frac{1}{\sqrt{|a|^2 + |b|^2}}, \text{ and } k_2 = \frac{1}{\sqrt{|a|^2 + |c|^2}}.$$

And, let the observables $\widehat{U}_1$, $\widehat{U}_2$, $\widehat{W}_1$, $\widehat{W}_2$ be defined as

$$\begin{aligned}
\widehat{U}_1 &= |u_1\rangle\langle u_1| \otimes \mathbb{I}, \\
\widehat{U}_2 &= \mathbb{I} \otimes |u_2\rangle\langle u_2|\,, \\
\widehat{W}_1 &= |w_1\rangle\langle w_1| \otimes \mathbb{I} = k_1^2\Big(|a|^2\,|v_1\rangle\langle v_1| + a\bar{b}\,|v_1\rangle\langle u_1| + \bar{a}b\,|u_1\rangle\langle v_1| + |b|^2\,|u_1\rangle\langle u_1|\Big), \\
\widehat{W}_2 &= \mathbb{I} \otimes |w_2\rangle\langle w_2| = k_2^2\Big(|a|^2\,|v_2\rangle\langle v_2| + a\bar{c}\,|v_2\rangle\langle u_2| + \bar{a}c\,|u_2\rangle\langle v_2| + |c|^2\,|u_2\rangle\langle u_2|\Big).
\end{aligned}$$

Then, the probability of obtaining the outcome $v_\psi(\widehat{U}_1) = 1$ and $v_\psi(\widehat{U}_2) = 1$ in a measurement of the physical quantities U_1 and U_2 in the state $|\psi\rangle$, that is, the probability of obtaining $v_\psi(\widehat{U}_1\widehat{U}_2) = 1$, is

$$P_\psi^{U_1U_2}(1) = a\,\widehat{U}_1\widehat{U}_2\,|v_1\rangle\,|v_2\rangle + b\,\widehat{U}_1\widehat{U}_2\,|u_1\rangle\,|v_2\rangle + c\,\widehat{U}_1\widehat{U}_2\,|v_1\rangle\,|u_2\rangle = 0.$$

Therefore, U_1 and U_2 cannot simultaneously have the value 1 in the state $|\psi\rangle$. Recall that $v_\psi(\widehat{Q})=0 \implies v_\psi\left(\widehat{Q}^{\complement}\right)=1$. Therefore, the probability of obtaining the outcome $v_\psi(\widehat{W}_1)=1$ given that $v_\psi(\widehat{U}_2)=0$, is equivalent to obtaining the outcome $v_\psi(\widehat{W}_1)=1$ given that $v_\psi\left(\widehat{U}_2^{\complement}\right)=1$. That is, $P_\psi^{W_1|U_2}(1\mid 0)$ is equivalent to $P_\psi^{W_1|U_2^{\complement}}(1\mid 1)$. Then,

$$P_\psi^{W_1|U_2}(1\mid 0)=P_\psi^{W_1|U_2^{\complement}}(1\mid 1)=\frac{\left\|\widehat{W}_1\widehat{U}_2^{\complement}|\psi\rangle\right\|^2}{\left\|\widehat{U}_2^{\complement}|\psi\rangle\right\|^2},$$

where

$$\widehat{U}_2^{\complement}|\psi\rangle=\left(\mathbb{I}-\widehat{U}_2\right)|\psi\rangle=|\psi\rangle-c|v_1\rangle|u_2\rangle=a|v_1\rangle|v_2\rangle+b|u_1\rangle|v_2\rangle,\ \text{and}\ \frac{1}{\left\|\widehat{U}_2^{\complement}|\psi\rangle\right\|^2}=\frac{1}{|a|^2+|b|^2}=k_1^2.$$

Therefore,

$$P_\psi^{W_1|U_2}(1\mid 0)=P_\psi^{W_1|U_2^{\complement}}(1\mid 1)=\left\|k_1\widehat{W}_1\widehat{U}_2^{\complement}|\psi\rangle\right\|^2=\left\|k_1\left(a|v_1\rangle|v_2\rangle+b|u_1\rangle|v_2\rangle\right)\right\|^2=1.$$

Therefore, if $v_\psi(\widehat{U}_2)=0$ then $v_\psi(\widehat{W}_1)=1$ in the state $|\psi\rangle$, that is, $v_\psi(\widehat{U}_2)=0 \implies v_\psi(\widehat{W}_1)=1$. Similarly, the probability of obtaining the outcome $v_\psi(\widehat{W}_2)=1$ given that $v_\psi(\widehat{U}_1)=0$, is equivalent to obtaining the outcome $v_\psi(\widehat{W}_2)=1$ given that $v_\psi\left(\widehat{U}_1^{\complement}\right)=1$. Therefore,

$$P_\psi^{W_2|U_1}(1\mid 0)=P_\psi^{W_2|U_1^{\complement}}(1\mid 1)=\frac{\left\|\widehat{U}_1^{\complement}\widehat{W}_2|\psi\rangle\right\|^2}{\left\|\widehat{U}_1^{\complement}|\psi\rangle\right\|^2},$$

where

$$\widehat{U}_1^{\complement}|\psi\rangle=\left(\mathbb{I}-\widehat{U}_1\right)|\psi\rangle=|\psi\rangle-b|u_1\rangle|v_2\rangle=a|v_1\rangle|v_2\rangle+c|v_1\rangle|u_2\rangle,\ \text{and}\ \frac{1}{\left\|\widehat{U}_1^{\complement}|\psi\rangle\right\|^2}=\frac{1}{|a|^2+|c|^2}=k_2^2.$$

Then,

$$P_\psi^{W_2|U_1}(1\mid 0)=P_\psi^{W_2|U_1^{\complement}}(1\mid 1)=\left\|k_2\widehat{U}_1^{\complement}\widehat{W}_2|\psi\rangle\right\|^2=\left\|k_2\left(a|v_1\rangle|v_2\rangle+c|v_1\rangle|u_2\rangle\right)\right\|^2=1.$$

Therefore, if $v_\psi(\widehat{U}_1)=0$ then $v_\psi(\widehat{W}_2)=1$ in the state $|\psi\rangle$, that is, $v_\psi(\widehat{U}_1)=0 \implies v_\psi(\widehat{W}_2)=1$. Finally, the probability of obtaining the outcome $v_\psi(\widehat{W}_1)=0$ and $v_\psi(\widehat{W}_2)=0$ in a measurement of the physical quantities W_1 and W_2 in the state $|\psi\rangle$ can be found by considering the complements of $\widehat{W}_1$ and $\widehat{W}_2$. If $v_\psi(\widehat{W}_1)=0$, then $v_\psi\left(\widehat{W}_1^{\complement}\right)=1$, and if $v_\psi(\widehat{W}_2)=0$, then $v_\psi\left(\widehat{W}_2^{\complement}\right)=1$. Then,

$$P_\psi^{W_1,W_2}(0,0)=P_\psi^{W_1^{\complement}W_2^{\complement}}(1)=\left\|\widehat{W}_1^{\complement}\widehat{W}_2^{\complement}|\psi\rangle\right\|^2=\left\|(\mathbb{I}-\widehat{W}_1-\widehat{W}_2+\widehat{W}_1\widehat{W}_2)|\psi\rangle\right\|^2.$$

To facilitate clarity, each of these terms will be considered in turn. The first term is

$$\widehat{W}_1|\psi\rangle=a|v_1\rangle|v_2\rangle+b|u_1\rangle|v_2\rangle+k_1^2c|a|^2|v_1\rangle|u_2\rangle+k_1^2\overline{a}bc|u_1\rangle|u_2\rangle.$$

The second term is

$$\widehat{W}_2|\psi\rangle=a|v_1\rangle|v_2\rangle+k_2^2b|a|^2|u_1\rangle|v_2\rangle+c|v_1\rangle|u_2\rangle+k_2^2\overline{a}bc|u_1\rangle|u_2\rangle.$$

The third term $\widehat{W}_1\widehat{W}_2\,|\psi\rangle$,

$$\widehat{W}_1\widehat{W}_2\,|\psi\rangle = k_1^2k_2^2|a|^2\left(a\,|v_1\rangle\,|v_2\rangle + b\,|u_1\rangle\,|v_2\rangle + c\,|v_1\rangle\,|u_2\rangle + \frac{\overline{a}bc}{|a|^2}\,|u_1\rangle\,|u_2\rangle\right).$$

Let $\left|\psi'\right\rangle = (\mathbb{I} - \widehat{W}_1 - \widehat{W}_2 + \widehat{W}_1\widehat{W}_2)\,|\psi\rangle$, then

$$\begin{aligned}\left|\psi'\right\rangle = a&\left(k_1^2k_2^2|a|^2 - 1\right)|v_1\rangle\,|v_2\rangle\\ &+ k_2^2|a|^2\,b\left(k_1^2 - 1\right)|u_1\rangle\,|v_2\rangle\\ &+ k_1^2|a|^2\,c\left(k_2^2 - 1\right)|v_1\rangle\,|u_2\rangle\\ &+ \overline{a}bc\left(k_1^2k_2^2 - k_1^2 - k_2^2\right)|u_1\rangle\,|u_2\rangle\,.\end{aligned}$$

The coefficient of the $|v_1\rangle\,|v_2\rangle$ term can be expressed in terms of a, b, and c by substituting in for k_1 and k_2 in the expression $a\left(k_1^2k_2^2|a|^2 - 1\right)$,

$$a\left(k_1^2k_2^2|a|^2 - 1\right) = a\left(\frac{|a|^2 - \left(|a|^2 + |b|^2\right)\left(|a|^2 + |c|^2\right)}{\left(|a|^2 + |b|^2\right)\left(|a|^2 + |c|^2\right)}\right) = \frac{-a|b|^2|c|^2}{|a|^2 + |b|^2|c|^2}.$$

Note that $k_1^2k_2^2 = {}^1\!/_{|a|^2+|b|^2|c|^2}$. Similarly, the coefficient of the term $|u_1\rangle\,|v_2\rangle$ in terms of a, b, and c is found by substituting in for k_1 and k_2 in the expression $k_2^2|a|^2\,b\left(k_1^2 - 1\right)$,

$$k_2^2|a|^2\,b\left(k_1^2 - 1\right) = \frac{|a|^2\,b}{|a|^2 + |b|^2|c|^2} - \frac{|a|^2\,b}{|a|^2 + |c|^2}.$$

By expanding second term with k_1^2/k_1^2 and factorising, the coefficient for the $|u_1\rangle\,|v_2\rangle$ term is

$$k_2^2|a|^2\,b\left(k_1^2 - 1\right) = \frac{|a|^2\,b|c|^2}{|a|^2 + |b|^2|c|^2},$$

By similar methods, the coefficient of the $|v_1\rangle\,|u_2\rangle$ term is

$$k_1^2|a|^2\,c\left(k_2^2 - 1\right) = \frac{|a|^2|b|^2\,c}{|a|^2 + |b|^2|c|^2}$$

And, the coefficient of the $|u_1\rangle\,|u_2\rangle$ term, is

$$\overline{a}bc\left(k_1^2k_2^2 - k_1^2 - k_2^2\right) = \overline{a}bc\left(\frac{1 - \left(|a|^2 + |c|^2\right) - \left(|a|^2 + |b|^2\right)}{|a|^2 + |b|^2|c|^2}\right) = \frac{-\overline{a}bc|a|^2}{|a|^2 + |b|^2|c|^2}.$$

Finally,

$$P_\psi^{W_1,W_2}(0,0) = P_\psi^{W_1^\complement W_2^\complement}(1) = \left\|\left|\psi'\right\rangle\right\|^2 = \frac{|a|^2|b|^2|c|^2}{|a|^2 + |b|^2|c|^2}.$$

In summary,

$$P_{\psi}^{U_1U_2}(1) = 0, \tag{21}$$
$$P_{\psi}^{W_1|U_2}(1 \mid 0) = 1, \tag{22}$$
$$P_{\psi}^{W_2|U_1}(1 \mid 0) = 1, \tag{23}$$
$$P_{\psi}^{W_1,W_2}(0,0) = \frac{|a|^2|b|^2|c|^2}{|a|^2 + |b|^2|c|^2}. \tag{24}$$

4.2 Goldstein's argument

The probabilities (22) and (23) show that $v_{\psi}(\widehat{U}_2) = 0 \implies v_{\psi}(\widehat{W}_1) = 1$ and $v_{\psi}(\widehat{U}_1) = 0 \implies v_{\psi}(\widehat{W}_2) = 1$. If $v_{\psi}(\widehat{U}_2) = 1$, then the valuation $v_{\psi}(\widehat{W}_1)$ is in the spectrum of $\widehat{W}_1$, where $\mathrm{spec}(\widehat{W}_1) = \{0,1\}$. Similarly for $v_{\psi}(\widehat{U}_1) = 1$, we have that $v_{\psi}(\widehat{W}_2) \in \{0,1\}$. Then, the possible configurations of hidden variables that follow from the implications are presented in the following table.

$v_{\psi}(\widehat{U}_1)$	$v_{\psi}(\widehat{U}_2)$	$v_{\psi}(\widehat{W}_1)$	$v_{\psi}(\widehat{W}_2)$
0	0	1	1
0	1	0 or 1	1
1	0	0	0 or 1
1	1	0 or 1	0 or 1

From (24) we know that we obtain the result $v_{\psi}(\widehat{W}_1) = v_{\psi}(\widehat{W}_2) = 0$ with a non-zero probability. However, the only possible configuration which allows $v_{\psi}(\widehat{W}_1) = v_{\psi}(\widehat{W}_2) = 0$ is when $v_{\psi}(\widehat{U}_1) = v_{\psi}(\widehat{U}_2) = 1$ which is in contradiction to (21). Therefore, there are no possible configuration of hidden variables which is able to reproduce the predictions of quantum mechanics. Notably, the argument holds for all states of the form $|\psi\rangle = a\,|v_1\rangle\,|v_2\rangle + b\,|u_1\rangle\,|v_2\rangle + c\,|v_1\rangle\,|u_2\rangle$, where $abc \neq 0$, and $|a|^2 + |b|^2 + |c|^2 = 1$, which, as we will show, is almost all entangled states.

4.3 The maximum probability of obtaining the contradiction

The values of $|a|$, $|b|$, and $|c|$ are positive real numbers, due to the constraint $abc \neq 0$. Therefore, the probability of obtaining $v_{\psi}(\widehat{W}_1) = 0$ and $v_{\psi}(\widehat{W}_2) = 0$, can be written as a function of the magnitudes of a, b, and c,

$$P_{\psi}^{W_1,W_2}(0,0) = f(x,y,z) = \frac{x^2y^2z^2}{x^2 + y^2z^2},$$

where $x = |a|$, $y = |b|$, and $z = |c|$ are positive real numbers, subject to the constraint

$$x^2 + y^2 + z^2 = 1.$$

By using the constraint $z^2 = 1 - x^2 - y^2$, one can create a 2-D contour plot of $P_{\psi}^{W_1,W_2}(0,0)$, see Figure 2. Note that the x and y axes represent the magnitude of the complex number a and b respectively.
The maximum probability of obtaining $v_{\psi}(\widehat{W}_1) = 0$ and $v_{\psi}(\widehat{W}_2) = 0$, can be expressed by the square of the magnitudes of a, b, and c, that is $l = |a|^2$, $m = |b|^2$, and $q = |z|^2$,

$$P_{\psi}^{W_1,W_2}(0,0) = \frac{lmn}{l + mn}.$$

By using the constraint $q = 1 - l - m$, a function $g(l,m)$ can be defined as

$$g(l,m) = \frac{lm(1-l-m)}{l + m(1-l-m)}.$$

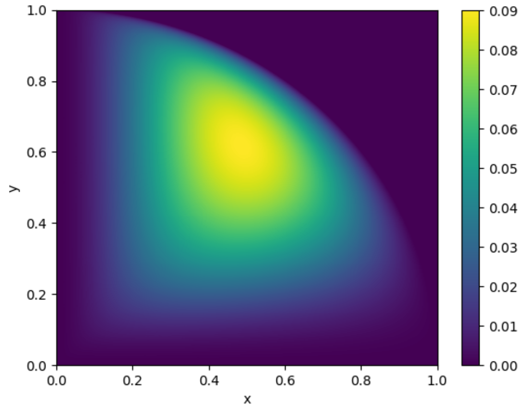

Figure 2: $P_{\psi}^{W_1,W_2}(0,0)$

The maximum of $g(l,m)$ is found by solving the system of equations

$$\frac{\partial}{\partial l}\, g(l,m) = \frac{m\left(l^2m - l^2 + 2lm^2 - 2lm + m^3 - 2m^2 + m\right)}{\left(l + m\left(1 - l - m\right)\right)^2} = 0,$$
$$\frac{\partial}{\partial m}\, g(l,m) = \frac{l\left(-2lm - l^2 + l\right)}{\left(l + m\left(1 - l - m\right)\right)^2} = 0.$$

The solution, for positive real numbers l and m is $l_{max} = \sqrt{5} - 2$, and $m_{max} = \frac{5\sqrt{5}-11}{2\left(\sqrt{5}-2\right)}$. Then, the maximum probability of obtaining $v_\psi(\widehat{W}_1) = 0$ and $v_\psi(\widehat{W}_2) = 0$ is

$$g(l_{max}, m_{max}) = \frac{1}{2}\left(5\sqrt{5} - 11\right) \approx 0.09,$$

which is the same as the probability calculated in Hardy's argument. The values for $|a_{max}|$, $|b_{max}|$, $|c_{max}|$, and $|\psi_{max}\rangle$ are therefore

$$|a_{max}| = \sqrt{\sqrt{5} - 2}, \quad |b_{max}| = \sqrt{\frac{5\sqrt{5} - 11}{2(\sqrt{5} - 2)}}, \quad |c_{max}| = \sqrt{\frac{3 - \sqrt{5}}{2}},$$
$$|\psi_{max}\rangle = a_{max}\,|v_1\rangle\,|v_2\rangle + b_{max}\,|u_1\rangle\,|v_2\rangle + c_{max}\,|v_1\rangle\,|u_2\rangle\,.$$

To find the Schmidt-decomposition of $|\psi_{max}\rangle$, define two new orthonormal bases $\mathcal{B}_{s1} = \left\{\left|e_1^1\right\rangle, \left|e_1^2\right\rangle\right\}$ and $\mathcal{B}_{s2} = \left\{\left|f_2^1\right\rangle, \left|f_2^2\right\rangle\right\}$ for particle 1 and particle 2 respectively, such that $\left|e_1^1\right\rangle = |u_1\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $\left|e_1^2\right\rangle = |v_1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, $\left|f_2^1\right\rangle = |u_2\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $\left|f_2^2\right\rangle = |v_2\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Then, the state that gives the maximum probability of obtaining $v_\psi(\widehat{W}_1) = 0$ and $v_\psi(\widehat{W}_2) = 0$ can be written as

$$|\psi_{max}\rangle = \sum_{i,j=1}^{2} C_{ij}\left|e_1^i\right\rangle\left|f_2^j\right\rangle, \text{ where } C = \begin{bmatrix} 0 & b_{max} \\ c_{max} & a_{max} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\frac{5\sqrt{5}-11}{2(\sqrt{5}-2)}} \\ \sqrt{\frac{3-\sqrt{5}}{2}} & \sqrt{\sqrt{5}-2} \end{bmatrix}.$$

The singular values of C are $\lambda_1 = \sqrt{\frac{1-\sqrt{-13+6\sqrt{5}}}{2}} \approx 0.4211$, and $\lambda_2 = \sqrt{\frac{1+\sqrt{-13+6\sqrt{5}}}{2}} \approx 0.9070$. Therefore, for a suitable choice of bases $\mathcal{B}_1 = \left\{|r_1\rangle, |s_1\rangle\right\}$ and $\mathcal{B}_2 = \left\{|r_2\rangle, |s_2\rangle\right\}$,

$$|\psi_{max}\rangle = 0.9070\,|r_1\rangle\,|r_2\rangle - 0.4211\,|s_1\rangle\,|s_2\rangle\,,$$

which is the same state as in Hardy's argument.

4.4 Considerations of the limitations of the argument

From the argument presented earlier in this thesis, it is shown that Goldstein's argument holds for any entangled state which can be written in the form

$$|\psi\rangle = a\,|v_1\rangle\,|v_2\rangle + b\,|u_1\rangle\,|v_2\rangle + c\,|v_1\rangle\,|u_2\rangle \text{ where } |a|^2 + |b|^2 + |c|^2 = 1 \text{ and } abc \neq 0. \tag{25}$$

Therefore, we will show that any entangled state that is not maximally entangled can be written in the form (25), thereby showing that Goldstein's argument holds for any entangled state that is not maximally entangled.

Let $\mathcal{B}_1 = \left\{ \left|e_1^1\right\rangle = |u_1\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left|e_1^2\right\rangle = |v_1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \right\}$ and $\mathcal{B}_2 = \left\{ \left|f_2^1\right\rangle = |u_2\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left|f_2^2\right\rangle = |v_2\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \right\}$ be orthonormal bases for particle 1 and particle 2 respectively. Then, a state of the form (25) can be expressed as

$$|\psi\rangle = \sum_{i,j=1}^{2} C_{ij} \left|e_1^i\right\rangle \left|f_2^j\right\rangle, \text{ where } C = \begin{bmatrix} 0 & b \\ c & a \end{bmatrix}.$$

By singular value decomposition $C = UDV$, and, for any entangled state the matrix of its singular values is

$$D_e = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \text{ where } \lambda_1, \lambda_2 \in \mathbb{R}_{>0} \text{ and } \lambda_1^2 + \lambda_2^2 = 1.$$

Then, an entangled state can be written in the form (25) if and only if there exists unitary matrices U and V such that $C = UD_eV$. Note that, $CC^* = UD_e^2U^*$ and $C^*C = VD_e^2V^*$, which means that CC^* and C^*C are both unitarily diagonalisable by the unitary matrices U and V respectively. Therefore, U and V are orthonormal eigenvector bases of CC^* and C^*C respectively, with eigenvalues equal to the square of the singular values. The eigenvectors of CC^* are found by solving $\left(CC^* - \lambda_i^2\mathbb{I}\right) w = 0$,

$$\begin{bmatrix} |b|^2 - \lambda_i^2 & \overline{a}b \\ a\overline{b} & |a|^2|c|^2 - \lambda_i^2 \end{bmatrix} w = 0 \implies w_i = \begin{pmatrix} -\overline{a}b/\left(|b|^2 - \lambda_i^2\right) \\ 1 \end{pmatrix}.$$

There are two distinct cases where Goldstein's argument fail, first, if $|b|^2 - \lambda_i^2 = 0$, and second, if $\lambda_1 = \lambda_2$, these deficiencies will be discussed soon. However, if $\lambda_1 \neq \lambda_2$, then there is a component of w_1 that is orthogonal to w_2, and therefore, for a choice of a and b such that $\langle w_1|w_2\rangle = 0$ and $\|w_1\| = \|w_2\| = 1$, there exists an orthonormal eigenvector basis for CC^*. By a similar argument, if $\lambda_1 \neq \lambda_2$ then, for choice of b and c such that $\langle \omega_1|\omega_2\rangle = 0$ and $\|\omega_1\| = \|\omega_2\| = 1$, there exists an eigenvector basis for C^*C,

$$\begin{bmatrix} |c|^2 - \lambda_i^2 & b\overline{c} \\ \overline{b}c & |a|^2|b|^2 - \lambda_i^2 \end{bmatrix} \omega = 0 \implies \omega_i = \begin{pmatrix} -b\overline{c}/\left(|c|^2 - \lambda_i^2\right) \\ 1 \end{pmatrix}.$$

To ensure that both CC^* and C^*C have orthogonal eigenvectors bases there must exist a, b, and c such that $\langle w_1|w_2\rangle = \langle \omega_1|\omega_2\rangle = 0$, then, the following system of equations must be satisfied,

$$\left(|b|^2 - \lambda_1^2\right)\left(|b|^2 - \lambda_2^2\right) = -|a|^2|b|^2, \quad \left(|c|^2 - \lambda_1^2\right)\left(|c|^2 - \lambda_2^2\right) = -|b|^2|c|^2, \quad |a|^2 + |b|^2 + |c|^2 = 1, \quad \lambda_1^2 + \lambda_2^2 = 1.$$

As an example, one set of solutions to the underdetermined system is

$$\lambda_1 = \frac{1}{2}\left(1 - \sqrt{2|c|^4 - 2|c|^2 + 1}\right), \; \lambda_2 = \frac{1}{2}\left(1 + \sqrt{2|c|^4 - 2|c|^2 + 1}\right), \; |a|^2 = \frac{1 - |c|^2}{2}, \; |b|^2 = \frac{1 - |c|^2}{2}, \; |c| \in (0, 1).$$

Now, c can be chosen such that $|c|^2 - \lambda_i^2 \neq 0$, in order to avoid the first deficiency which was mentioned, and similarly it is possible to ensure $|b|^2 - \lambda_i^2 \neq 0$, since the system of equations is underdetermined. Then it is shown that there exists orthogonal eigenvector bases for both CC^* and C^*C. To ensure that the bases are orthonormal, the eigenvectors can be scaled such that $\|\omega_1\| = \|\omega_2\| = 1$. The second deficiency mentioned earlier occurs in maximally entangled states, where the singular values are equal, that is, when $\lambda_1 = \lambda_2$. Then, the eigenvectors w_1 and w_2 are parallel, and therefore, they do not form an eigenvector basis for CC^*. Similarly, ω_1 and ω_2 are parallel, which means they do not form an eigenvector basis for C^*C. Consequently, maximally entangled states cannot be written in the form (25). Therefore, it is shown that any entangled state, except for maximally entangled states, can be written in the form (25), and thus admits a Hardy-type argument.

5 Conclusion and outlook

In this thesis, we have presented a detailed derivation of the arguments put forward by Lucien Hardy [8][9] and Sheldon Goldstein [10]. We clarified these arguments by distinguishing clearly between the quantum mechanical framework and the hidden-variables model, which previous authors have left implicit. The contradiction emerging from these arguments was demonstrated using the valuation function. The valuation function, if required to satisfy the mild conditions (SPEC) and (FUNC), shows that there is no consistent assignment of pre-existing values to the physical quantities which satisfy the criterion of reality, in the given quantum state. Additionally, we assumed faithful measurements and that the principle of locality holds. The resulting contradiction demonstrates that any hidden-variables models making the same assumptions will inevitably encounter a contradiction, thereby ruling it out as a description of physical reality. We have also discussed the limitations of these arguments, including pointing out that the observables commute in Hardy's extension to the original argument, which the previous authors have not mentioned. We have substantiated the assertion that the argument retains validity for all pure entangled states, with the exception of maximally entangled ones, where our presentation aims to be easier to follow compared to the presentations offered by previous authors.

As noted, a previous author has shown that it is not possible to construct a Hardy-type argument using projective measurements. However, investigating whether positive operator-valued measures might admit a similar argument could be an intriguing pursuit. Moreover, examining different interpretations of realism and locality and observing their impacts on the argument and constraints on hidden-variables models could prove enlightening. Another interesting aspect of the argument is its logical structure; Hardy's argument involves implications, that is, values of physical observables depending on values of other physical observables with probability equal to unity. It might be interesting to investigate this logical structure and identify areas of discordance with classical logic. Ultimately, Hardy's argument sheds light on the non-classical aspects of quantum mechanics, which still brims with foundational questions awaiting exploration.

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Appendix

A - Plot of $P\left(v_\psi\left(\widehat{W}_1=0\right), v_\psi\left(\widehat{W}_2=0\right)\right)$.

```
import matplotlib.pyplot as plt
from numpy import linspace, sqrt, zeros


def f(a, b):
    c = sqrt(abs(1 - a**2 - b**2))
    if a**2 + b**2 + c**2 > 1:
        return 0

    if a * b * c == 0:
        return 0

    return (a**2 * b**2 * c**2) / (a**2 + b**2 * c**2)


A = linspace(0, 1, 1000)
B = linspace(0, 1, 1000)
values = zeros((len(A), len(B)))
highest_value = 0
for x, a in enumerate(A):
    for y, b in enumerate(B):
        values[y, x] = f(a, b)
        if values[y, x] > highest_value:
            highest_value = values[y, x]

plt.figure(1)
plt.xlabel("x")
plt.ylabel("y")
plt.title("")
plt.contourf(A, B, values, 1000)
plt.colorbar(ticks=[round(i, 3) for i in linspace(0, highest_value, 10)])
plt.show()
```

B - Goldstein's argument fails in maximally entangled states.

Let $\mathcal{B}_1 = \left\{\left|e_1^1\right\rangle = |u_1\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left|e_1^2\right\rangle = |v_1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right\}$ and $\mathcal{B}_2 = \left\{\left|f_2^1\right\rangle = |u_2\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left|f_2^2\right\rangle = |v_2\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right\}$ be orthonormal bases for particle 1 and particle 2 respectively. Then, a state of the form

$$|\psi\rangle = a\,|v_1\rangle\,|v_2\rangle + b\,|u_1\rangle\,|v_2\rangle + c\,|v_1\rangle\,|u_2\rangle \text{ where } |a|^2 + |b|^2 + |c|^2 = 1 \text{ and } abc \neq 0, \tag{26}$$

can be expressed as

$$|\psi\rangle = \sum_{i,j=1}^{2} C_{ij} \left|e_1^i\right\rangle \left|f_2^j\right\rangle, \text{ where } C = \begin{bmatrix} 0 & b \\ c & a \end{bmatrix}.$$

For maximally entangled states, the singular values are equal, and therefore the matrix of singular values is

$$D_m = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda \mathbb{I}.$$

Therefore, a maximally entangled state can be written in the form (26) if and only if there exists unitary matrices U and V such that $C = UD_mV$. If we assume that there exists unitary matrices U and V that satisfies the condition, then

$$CC^* = UD_m D_m^* U^* = |\lambda|^2 \mathbb{I}, \text{ which contradicts } CC^* = \begin{bmatrix} |b|^2 & \overline{a}b \\ a\overline{b} & |a|^2|c|^2 \end{bmatrix},$$
$$\text{where } a, b, c \in \mathbb{C} \text{ such that } |a|^2 + |b|^2 + |c|^2 = 1 \text{ and } abc \neq 0.$$

Therefore, there are no unitary matrices U and V which satisfies $C = UD_m V$, and consequently maximally entangled states cannot be written in the form (26) and the procedure used in Goldstein's argument is therefore not applicable.