The algebra of observables in noncommutative deformation theory

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THE ALGEBRA OF OBSERVABLES IN NONCOMMUTATIVE DEFORMATION THEORY

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ABSTRACT. We consider the algebra $\mathcal{O}(\mathsf{M})$ of observables and the (formally) versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ defined by the noncommutative deformation functor $\mathsf{Def}_{\mathsf{M}}$ of a family $\mathsf{M} = \{M_1, \dots, M_r\}$ of right modules over an associative k-algebra A. By the Generalized Burnside Theorem, due to Laudal, η is an isomorphism when A is finite dimensional, M is the family of simple A-modules, and k is an algebraically closed field. The purpose of this paper is twofold: First, we prove a form of the Generalized Burnside Theorem that is more general, where there is no assumption on the field k. Secondly, we prove that the $\mathcal{O}$-construction is a closure operation when A is any finitely generated k-algebra and M is any family of finite dimensional A-modules, in the sense that $\eta_B : B \to \mathcal{O}^B(\mathsf{M})$ is an isomorphism when $B = \mathcal{O}(\mathsf{M})$ and M is considered as a family of B-modules.

1. Introduction

Let k be a field, let A be a finite dimensional associative algebra over k, and let $\mathsf{M} = \{M_1, \dots, M_r\}$ be the family of simple right A-modules, up to isomorphism. We consider the algebra homomorphism

$$\rho : A \to \bigoplus_{i=1}^{r} \operatorname{End}_k(M_i)$$

given by right multiplication of A on the family M. By the extended version of the classical Burnside Theorem, ρ is surjective when k is algebraically closed, and if A is semisimple, then it is an isomorphism. We remark that Artin-Wedderburn theory gives a version of the theorem that holds over any field:

Theorem (Classical Burnside Theorem)**.** *Let A be a finite dimensional k-algebra, and let $\{M_1, \dots, M_r\}$ be the family of simple right A-modules. If $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$, then $\rho : A \to \oplus_i \operatorname{End}_k(M_i)$ is surjective.*

In Laudal [3], a generalization called the Generalized Burnside Theorem was obtained. This is a structural result for not necessarily semisimple algebras, and the essential idea of Laudal was to replace ρ with the versal morphism η defined by noncommutative deformations of modules. Let us recall the construction:

Let A be an arbitrary associative k-algebra, let $\mathsf{M} = \{M_1, \dots, M_r\}$ be a family of right A-modules, and consider the noncommutative deformation functor $\mathsf{Def}_{\mathsf{M}}$. This functor has a pro-representing hull H and a versal family M_H if M is a swarm. Following Laudal [3], we define the *algebra of observables* of a swarm M to be $\mathcal{O}(\mathsf{M}) = \operatorname{End}_H(M_H) \cong (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$, and its *versal morphism* to be the

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algebra homomorphism $\eta : A \to \mathcal{O}(\mathsf{M})$ given by right multiplication of A on the versal family M_H. It fits into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) \\ & \searrow^{\rho} & \downarrow \\ & & \oplus_{i=1}^r \operatorname{End}_k(M_i) \end{array}$$

where $\rho : A \to \oplus_{i=1}^r \operatorname{End}_k(M_i)$ is the algebra homomorphism given by right multiplication of A on the family M. By Theorem 1.2 in Laudal [3], it follows that η is an isomorphism when A is finite dimensional, M is the family of simple A-modules, and k is algebraically closed. In this paper, we prove a more general version of this result:

Theorem (Generalized Burnside Theorem)**.** *Let A be a finite dimensional k-algebra, and let M be the family of simple right A-modules, up to isomorphism. The versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is injective. If $\operatorname{End}_A(M_i) = k$ for $1 \leq i \leq r$, then η is an isomorphism. In particular, η is an isomorphism if k is algebraically closed.*

In case $D_i = \operatorname{End}_A(M_i)$ is a division algebra with $\dim_k D_i > 1$ for some simple module M_i, it is often not difficult to describe the image of η as a subalgebra of $\mathcal{O}(\mathsf{M})$, and we shall give examples. As an application of the theorem, we introduce the standard form of any finite dimensional algbra A, given as

$$A \cong \mathcal{O}(\mathsf{M}) = (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j))$$

when $\operatorname{End}_A(M_i) = k$ for $1 \leq i \leq r$, or as a subalgebra of $\mathcal{O}(\mathsf{M})$ in general.

Let A be any finitely generated k-algebra and let M be any family of finite dimensional right A-modules. In this more general situation, the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is not necessarily an isomorphism. However, we may consider the algebra $B = \mathcal{O}(\mathsf{M})$ of observables, and M as a family of right B-modules, and iterate the process. We prove that the operation $(A, \mathsf{M}) \mapsto (B, \mathsf{M})$ has the following *closure property*:

Theorem (Closure Property)**.** *Let A be a finitely generated k-algebra, let M be a family of finite dimensional A-modules, and let $B = \mathcal{O}(\mathsf{M})$. Then the versal morphism $\eta^B : B \to \mathcal{O}^B(\mathsf{M})$ of M, considered as a family of right B-modules, is an isomorphism.*

One may consider a noncommutative algebraic geometry where the closed points are represented by simple modules; see for instance Laudal [4]. With this point of view, one may use versal morphisms $\eta : A \to \mathcal{O}(\mathsf{M})$ for families M of A-modules to construct noncommutative localization homomorphisms $\eta_s : A \to A_s$ for any $s \in A$. We explain this construction in Section 6. These localization maps are universal S-inverting localization maps, where $S = \{1, s, s^2, \dots \}$, and can be used as an essential building block for structure sheaves on noncommutative schemes.

2. Noncommutative deformations of modules

Let A be an associative algebra over a field k. For any right A-module M, there is a *deformation functor* $\mathsf{Def}_M : \mathsf{l} \to \mathsf{Sets}$ defined on the category l of commutative Artinian local k-algebras R with residue field k. We recall that $\mathsf{Def}_M(R)$ is the set of equivalence classes of pairs (M_R, τ_R), where M_R is an R-flat R-A bimodule on which k acts centrally, and $\tau_R : k \otimes_R M_R \to M$ is an isomorphism of right A-modules. Deformations in $\mathsf{Def}_M(R)$ are called *commutative deformations* since the base ring R is commutative.

Noncommutative deformations were introduced in Laudal [3]. The deformations considered by Laudal are defined over certain noncommutative base rings instead of the commutative base rings in l. In what follows, we shall give a brief account of noncommutative deformations of modules. We refer to Laudal [3], Eriksen [2] and Eriksen, Laudal, Siqveland [1] for further details.

For any positive integer r and any family $\mathsf{M} = \{M_1, \dots, M_r\}$ of right A-modules, there is a *noncommutative deformation functor* $\mathsf{Def}_{\mathsf{M}} : \mathsf{a}_r \to \mathsf{Sets}$, defined on the category a_r of noncommutative Artinian r-pointed k-algebras with exactly r simple modules (up to isomorphism). We recall that an r-pointed k-algebra R is one fitting into a diagram of rings $k^r \to R \to k^r$, where the composition is the identity. The condition that R has exactly r simple modules holds if and only if $\overline{R} \cong k^r$, where $\overline{R} = R/J(R)$ and $J(R)$ denotes the Jacobson radical of R.

The noncommutative deformations in $\mathsf{Def}_{\mathsf{M}}(R)$ are equivalence classes of pairs (M_R, τ_R), where M_R is an R-flat R-A bimodule on which k acts centrally, and $\tau_R : k^r \otimes_R M_R \to M$ is an isomorphism of right A-modules with $M = M_1 \oplus \dots \oplus M_r$. In concrete terms, an algebra R in a_r is a matrix ring $R = (R_{ij})$ with $R_{ij} = e_i R e_j$. By abuse of notation, we write e_i for the idempotent $e_i = (0, 0, \dots, i, \dots, 0)$ in k^r, and also for its image in R via the structural map $k^r \to R$. As left R-modules, we have that $M_R \cong (R_{ij} \otimes_k M_j)$ and its right A-module structure is given by an algebra homomorphism

$$\eta_R : A \to \mathrm{End}_R(M_R) \cong (R_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j))$$

that lifts $\rho : A \to \oplus_i \mathrm{End}_k(M_i)$. Explicitly, we interpret $\eta_R(a)$ as a right action of a on M_R via

$$\eta_R(a) = \sum_i e_i \otimes \rho_i + \sum_{i,j,l} r_{ij}^l \otimes \phi_{ij}^l \quad \iff \quad (e_i \otimes m_i)a = e_i \otimes (m_i a) + \sum_{j,l} r_{ij}^l \otimes \phi_{ij}^l(m_i)$$

where $\rho_i : A \to \mathrm{End}_k(M_i)$ is the algebra homomorphism given by the right action of A on M_i, such that $\rho = (\rho_1, \dots, \rho_r)$, and where $r_{ij}^l \in R_{ij}$ and $\phi_{ij}^l \in \mathrm{Hom}_k(M_i, M_j)$. Deformations in $\mathsf{Def}_{\mathsf{M}}(R)$ can therefore be represented by commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_R\ } & (R_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \\ & \searrow^{\rho} & \downarrow \\ & & \oplus_{i=1}^r \mathrm{End}_k(M_i) \end{array}$$

These deformations are called *noncommutative deformations* since the base ring R is noncommutative.

For any r-pointed algebra R, with structural maps $k^r \to R \to k^r$, we write $I(R) = \ker(R \to k^r)$. Recall that the pro-category $\widehat{\mathsf{a}}_r$ is the full subcategory of the category of r-pointed algebras consisting of algebras R such that $R/I(R)^n$ is Artinian for all n and such that R is complete in the $I(R)$-adic topology.

The family $\mathsf{M} = \{M_1, \dots, M_r\}$ is called a *swarm* if $\dim_k \mathrm{Ext}_A^1(M, M)$ is finite. In this case, the noncommutative deformation functor $\mathsf{Def}_{\mathsf{M}}$ has a pro-representing hull H in the pro-category $\widehat{\mathsf{a}}_r$ and a versal family $M_H \in \mathsf{Def}_{\mathsf{M}}(H)$; see Theorem 3.1 in Laudal [3]. The defining property of the miniversal pro-couple (H, M_H) is that the induced natural transformation

$$\phi : \mathrm{Mor}(H, -) \to \mathsf{Def}_{\mathsf{M}}$$

on a_r is smooth (which implies that ϕ_R is surjective for any R in a_r), and that ϕ_R is an isomorphism when $J(R)^2 = 0$. The miniversal pro-couple (H, M_H) is unique up to (non-canonical) isomorphism.

Let M be a swarm of right A-modules, and let (H, M_H) be the miniversal pro-couple of the noncommtutative deformation functor $\mathsf{Def}_{\mathsf{M}} : \mathsf{a}_r \to \mathsf{Sets}$. We define the *algebra of observables* of M to be

$$\mathcal{O}(\mathsf{M}) = \mathrm{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \mathrm{Hom}_k(M_i, M_j))$$

where $\widehat{\otimes}$ is the completed tensor product (the completion of the tensor product), and write $\eta : A \to \mathcal{O}(\mathsf{M})$ for the induced *versal morphism*, giving the right A-module structure on M_H. By construction, it fits into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & (H_{ij} \widehat{\otimes}_k \mathrm{Hom}_k(M_i, M_j)) \\ & \searrow^{\rho} & \downarrow \\ & & \oplus_{i=1}^{r} \mathrm{End}_k(M_i) \end{array}$$

Remark 1. *Notice that the diagram extends the right action of A on the family M to a right action of $\mathcal{O}(\mathsf{M})$, such that M is a family of right $\mathcal{O}(\mathsf{M})$-modules.*

Remark 2. *For any R in a_r and any deformation $M_R \in \mathsf{Def}_{\mathsf{M}}(R)$, there is a morphism $u : H \to R$ in $\widehat{\mathsf{a}}_r$ such that $\mathsf{Def}_{\mathsf{M}}(u)(M_H) = M_R$ by the versal property, and the deformation M_R is therefore given by the composition $\eta_R = u^* \circ \eta$ in the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & \mathcal{O}(\mathsf{M}) \\ & \searrow^{\eta_R} & \downarrow{\scriptstyle u^* = u \otimes \mathrm{id}} \\ & & (R_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \end{array}$$

In this sense, the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ determines all noncommutative deformations of the family M.

3. Iterated extensions and injectivity of the versal morphism

Let E be a right A-module and let $r \geq 1$ be a positive integer. If E has a *cofiltration* of length r, given by a sequence

$$E = E_r \xrightarrow{f_r} E_{r-1} \to \cdots \to E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 = 0$$

of surjective right A-module homomorphisms $f_i : E_i \to E_{i-1}$, then we call E an *iterated extension* of the right A-modules $M_1, M_2, \dots M_r$, where $M_i = \ker(f_i)$. In fact, the cofiltration induces short exact sequences

$$0 \to M_i \to E_i \xrightarrow{f_i} E_{i-1} \to 0$$

for $1 \leq i \leq r$. Hence $E_1 \cong M_1$, E_2 is an extension of E_1 by M_2, and in general, E_i is an extension of E_{i-1} by M_i.

Let $\mathsf{M} = \{M_1, \dots, M_r\}$ be a swarm of right A-modules, and let $\mathsf{Def}_{\mathsf{M}} : \mathsf{a}_r \to \mathsf{Sets}$ be its noncommutative deformation functor. Then $\mathsf{Def}_{\mathsf{M}}$ has a miniversal pro-couple (H, M_H), and we consider the induced versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ and its kernel $K = \ker(\eta)$.

We note that Theorem 3.2 in Laudal [3] holds without assumptions on the base field k, since the construction that precedes this theorem works over any field. From this observation, we obtain the following lemma:

Lemma 3. *Let M be a swarm of right A-modules. For any iterated extension E of the family M, we have that $E \cdot K = 0$.*

Let A be a finite dimensional k-algebra and let M be the family of all simple right A-modules, up to ismorphism. Then M is a swarm, and we may consider the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$. If k is algebraically closed, then the versal morphism η is injective by Corollary 3.1 in Laudal [3]. Using Lemma 3, we generalize this result:

Proposition 4. *If A, considered as a right A-module, is an iterated extension of a swarm M, then the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is injective. In particular, η is injective when A is a finite dimensional algebra and M is the family of simple right A-modules.*

Proof. If A is an iterated extension of M, then $1 \cdot K = 0$ by Lemma 3, and this implies that $K = 0$. If A is finite dimensional, then the right A-module A has finite length, and it is an iterated extension of the simple modules. $\square$

We remark that our proof, based on Lemma 3, holds whenever there is an element $e \in E$ such that $a \mapsto e{\cdot}a$ defines an injective right A-module homomorphism $A \to E$. This means that $\eta : A \to \mathcal{O}(\mathsf{M})$ is injective if there is an iterated extension E of M such that E contains a copy of A_A.

4. The Generalized Burnside Theorem

Let A be a finite dimensional k-algebra, and let $\mathsf{M} = \{M_1, \dots, M_r\}$ be the family of simple right A-modules, up to isomorphism. Then M is a swarm, and we consider the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) \\ & \searrow^{\rho} & \downarrow \\ & & \oplus_{i=1}^{r} \operatorname{End}_k(M_i) \end{array}$$

Clearly, ρ factors through $A/\operatorname{J}(A)$, and if $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$, then $A/\operatorname{J}(A) \to \oplus_i \operatorname{End}_k(M_i)$ is an isomorphism by the Artin-Wedderburn theory for semisimple algebras. This proves the Classical Burnside Theorem mentioned in the introduction. By Theorem 3.4 in Laudal [3], the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is an isomorphism when k is algebraically closed. We generalize this result:

Theorem 5. *Let A be a finite dimensional k-algebra and let M be the family of simple right A-modules, up to isomorphism. Then $\eta : A \to \mathcal{O}(\mathsf{M})$ is injective, and it is an isomorphism if $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$. In particular, the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is an isomorphism if k is algebraically closed.*

Proof. By Proposition 4, the versal morphism η is injective, and it is enough to prove that η is surjective when $\mathrm{End}_A(M_i) = k$ for $1 \le i \le r$. Note that η maps the Jacobson radical $\mathrm{J}(A)$ of A to the Jacobson radical $J = (\mathsf{J}(H)_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j))$ of $\mathcal{O}(\mathsf{M})$. Moreover, A is $\mathrm{J}(A)$-adic complete since it is finite dimensional, and $\mathcal{O}(\mathsf{M})$ is clearly J-adic complete. By a standard result for filtered algebras, it is therefore sufficient to show that $\mathrm{gr}_1(\eta) : J(A)/J(A)^2 \to J/J^2$ is surjective, since $\mathrm{gr}_0(\eta) : A/J(A) \to \oplus_i \mathrm{End}_k(M_i)$ is an isomorphism by the Classical Burnside Theorem. We notice that

$$J/J^2 \cong ((J(H)/J(H)^2)_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \cong (\mathrm{Ext}^1_A(M_i, M_j)^* \otimes_k \mathrm{Hom}_k(M_i, M_j))$$

since $J(H)/J(H)^2$ is the dual of the tangent space $(\mathrm{Ext}^1_A(M_i, M_j))$ of $\mathsf{Def}_{\mathsf{M}}$. We note that Lemma 3.7 in Laudal [3] holds over any field. Hence the map

$$J(A)/J(A)^2 \to (\mathrm{Ext}^1_A(M_i, M_j)^* \otimes_k \mathrm{Hom}_k(M_i, M_j))$$

induced by η is an isomorphism, and this completes the proof. □

5. The closure property

Let A be a finitely generated k-algebra of the form $A = k\langle x_1, \dots x_d\rangle/I$, and let $\mathsf{M} = \{M_1, \dots, M_r\}$ be a family of finite dimensional right A-modules. Then M is a swarm, since

$$\dim_k \mathrm{Ext}^1_A(M_i, M_j) \le \dim_k \mathrm{Der}_k(A, \mathrm{Hom}_k(M_i, M_j)) \le \dim_k \mathrm{Hom}_k(M_i, M_j)^d$$

The last inequality follows from the fact that any derivation $D : A \to \mathrm{Hom}_k(M_i, M_j)$ is determined by $D(x_l) \in \mathrm{Hom}_k(M_i, M_j)$ for $1 \le l \le d$. We consider the algebra of observables $B = \mathcal{O}(\mathsf{M})$ of the swarm M, and write $\eta : A \to B$ for its versal morphism. In general, $\mathsf{M} = \{M_1, \dots, M_r\}$ is a family of right B-modules via η.

Lemma 6. *The family $\mathsf{M} = \{M_1, \dots, M_r\}$ of right B-modules is the simple right B-modules, and it is swarm of B-modules.*

Proof. It follows from the Artin-Wedderburn theory that $\mathsf{M} = \{M_1, \dots, M_r\}$ is the family of simple modules over

$$\overline{B} = B/J(B) \cong (H/J(H) \otimes_k \mathrm{Hom}_k(M_i, M_j)) \cong \bigoplus_i \mathrm{End}_k(M_i).$$

Since B and $\overline{B} = B/J(B)$ have the same simple modules, it follows that M is the family of simple right B-modules. We have that $\mathrm{Ext}^1_B(M_i, M_j)$ is a quotient of $\mathrm{Der}_k(B, \mathrm{Hom}_k(M_i, M_j))$, and any derivation $D : B \to \mathrm{Hom}_k(M_i, M_j)$ satisfies $D(J^2) = JD(J) + D(J)J = 0$ when $J = J(B)$ since M is the family of simple B-modules. From the fact that

$$B/J^2 \cong ((H/\,\mathrm{J}(H)^2)_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j))$$

is finite dimensional, and in particular a finitely generated k-algebra, it follows from the argument preceding the lemma that M is a swarm of B-modules. □

In this situation, we may iterate the process. Since M is a swarm of right B-modules, the noncommutative deformation functor $\mathsf{Def}^B_{\mathsf{M}}$ of M, considered as a family of right B-modules, has a miniversal pro-couple (H^B, M^B_H). We write $\mathcal{O}^B(\mathsf{M}) = \mathrm{End}_{H^B}(M^B_H) \cong (H^B_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j))$ for its algebra of observables and $\eta^B : B \to \mathcal{O}^B(\mathsf{M})$ for its versal morphism.

Theorem 7. *Let A be a finitely generated k-algebra, let $\mathsf{M} = \{M_1, \dots, M_r\}$ be a family of finite dimensional A-modules, and let $B = \mathcal{O}(\mathsf{M})$. Then the versal morphism $\eta^B : B \to \mathcal{O}^B(\mathsf{M})$ of M, considered as a family of right B-modules, is an isomorphism.*

Proof. Since M is a swarm of A-modules and of B-modules, we may consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta} & B = \mathcal{O}(\mathsf{M}) & \xrightarrow{\eta^B} & C = \mathcal{O}^B(\mathsf{M}) \\ & {\scriptstyle\rho}\searrow & \downarrow & \swarrow & \\ & & \bigoplus_i \operatorname{End}_k(M_i) & & \end{array}$$

The algebra homomorphism η^B induces maps $B/\operatorname{J}(B)^n \to C/\operatorname{J}(C)^n$ for all $n \geq 1$, and it is enough to show that each of these induced maps is an isomorphism. For $n = 1$, we have

$$B/J(B) \cong C/J(C) \cong \bigoplus_i \operatorname{End}_k(M_i)$$

so it is clearly an isomorphism for $n = 1$. For $n \geq 2$, we have that $B_n = B/J(B)^n$ is a finite dimensional algebra with the same simple modules as B since $M_i J^n = 0$. We may therefore consider the versal morphism of the swarm M of right B_n-modules, which is an isomorphism by the Generalized Burnside Theorem since $\operatorname{End}_B(M_i) = k$ for $1 \leq i \leq r$. Finally, any derivation $D : B \to \operatorname{Hom}_k(M_i, M_j)$ satisfies $D(J^n) = 0$ when $n \geq 2$. Therefore, we have that

$$\operatorname{Ext}^1_{B_n}(M_i, M_j) \cong \operatorname{Ext}^1_B(M_i, M_j)$$

and this implies that $B/J(B)^n \to C/J(C)^n$ coincides with the versal morphism of the swarm M of right B_n-modules. It is therefore an isomorphism. □

Theorem 7 implies that the assignment $(A, \mathsf{M}) \mapsto (B, \mathsf{M})$ is a closure operation when A is a finitely generated k-algebra and $\mathsf{M} = \{M_1, \dots, M_r\}$ is a family of finite dimensional right A-modules. In other words, the algebra $B = \mathcal{O}(\mathsf{M})$ has the following properties:

(1) The family M is the family of simple right B-modules.
(2) The family M has exactly the same module-theoretic properties, in terms of extensions and matric Massey products, considered as a family of B-modules and as a family of A-modules.

Moreover, these properties characterize the algebra of observables $B = \mathcal{O}(\mathsf{M})$.

Remark 8. *Assume that k is a field that is not algebraically closed. When A is a finite dimensional k-algebra and M is the family of simple right A-modules, it could happen that the division algebra $D_i = \operatorname{End}_A(M_i)$ has dimension $\dim_k D_i > 1$ for some simple A-modules M_i. In this case, $\eta : A \to \mathcal{O}(\mathsf{M})$ is not necessarily an isomorphism. However, if the subfamily $\mathsf{M}' = \{M_i : \operatorname{End}_A(M_i) = k\} \subseteq \mathsf{M}$ is non-empty, we may consider the algebra $B = \mathcal{O}(\mathsf{M}')$, and it follows from the closure property that $\eta : B \to \mathcal{O}^B(\mathsf{M}')$ is an isomorphism. This means that the Generalized Burnside Theorem holds for the family M' of right B-modules.*

6. Noncommutative localizations via the algebra of observables

Let A be a finitely generated k-algebra, and denote by $X = \mathrm{Simp}(A)$ the set of (isomorphism classes of) simple finite dimensional right A-modules. For any $s \in A$, we write

$$D(s) = \{M \in X : M \xrightarrow{\cdot s} M \text{ is invertible}\} \subseteq X.$$

We note that $\{D(s)\}_{s \in A}$ is a base for a topology on X, since $D(s) \cap D(t) = D(st)$, which we call the *Jacobson topology* on $X = \mathrm{Simp}(A)$.

For any inclusion $\mathsf{M} \subseteq \mathsf{M}'$ of finite subsets of $D(s)$, there is a surjective algebra homomorphism $\mathcal{O}(\mathsf{M}') \to \mathcal{O}(\mathsf{M})$. We may consider the algebra homomorphism

$$\eta_s : A \to \varprojlim_{\mathsf{M} \subseteq D(s)} \mathcal{O}(\mathsf{M})$$

where the projective limit is taken over all finite subsets $\mathsf{M} \subseteq D(s)$. Notice that $\eta_s(s)$ is a unit, since it is a unit in $\mathcal{O}(\mathsf{M})$ for any finite subset $\mathsf{M} \subseteq D(s)$. We define A_s to be the subring of the projective limit

$$\varprojlim_{\mathsf{M} \subseteq D(s)} \mathcal{O}(\mathsf{M})$$

generated by $\eta_s(A)$ and $\eta_s(s)^{-1}$. By abuse of notation, we write η_s for the algebra homomorphism $\eta_s : A \to A_s$ into the subring A_s.

Let S be the multiplicative subset $S = \{1, s, s^2, \dots\} \subseteq A$. Then $\eta_s : A \to A_s$ is an S-inverting algebra homomorphism, and it has the following universal property: If $\phi : A \to B$ is any S-inverting algebra homomorphism, then there is a unique algebra homomorphism $\phi_s : A_s \to B$ such that $\phi_s \circ \eta_s = \phi$. We remark that A_s is a finitely generated k-algebra, generated by the images of the generators of A and $\eta_s(s)^{-1}$. In general, it is not a (left or right) ring of fractions.

7. Applications

Let A be a finite dimensional k-algebra. We consider the family $\mathsf{M} = \{M_1, \dots, M_r\}$ of simple right A-modules. By the Generalized Burnside Theorem, A can be written in *standard form* as

$$A \cong \mathrm{im}(\eta) \subseteq (H_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) = \mathcal{O}(\mathsf{M})$$

If $\mathrm{End}_A(M_i) = k$ for $1 \le i \le r$, then the standard form of A is $A \cong \mathcal{O}(\mathsf{M})$, and in general, it is a subalgebra of $\mathcal{O}(\mathsf{M})$.

The standard form can, for instance, be used to compare finite dimensional algebras and determine when they are isomorphic. Let us illustrate this with a simple example. Let k be a field, and let $A = k[G]$ be the group algebra of $G = \mathbb{Z}_3$. In concrete terms, we have that $A \cong k[x]/(x^3 - 1)$, and over a fixed algebraic closure $\overline{k}$ of k, we have that

$$x^3 - 1 = (x-1)(x^2+x+1) = (x-1)(x-\omega)(x-\omega^2)$$

with $\omega \in \overline{k}$. If $\mathrm{char}(k) \neq 3$ and $\omega \in k$, then the simple A-modules are given by $\mathsf{M} = \{M_0, M_1, M_2\}$, where $M_i = A/(x - \omega^i)$. Furthermore, a calculation shows that $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for $0 \le i, j \le 2$. Hence, the noncommutative deformation functor $\mathsf{Def}_{\mathsf{M}}$ has a pro-representing hull $H = k^3$ (it is rigid), and the versal morphism $\eta : A \to \mathcal{O}(\mathsf{M})$ is an isomorphism. The standard form of A is therefore given by

$$A = k[\mathbb{Z}_3] \cong k^3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

If $\mathrm{char}(k) = 3$, then M_0 is the only simple A-module since $x^3 - 1 = (x-1)^3$, and we find that $\mathrm{Ext}^1_A(M_0, M_0) = k$. In this case, it turns out that $H \cong k[[t]]/(t^3)$, and the standard form of A is given by $A = k[\mathbb{Z}_3] \cong k[t]/(t^3)$. In both cases, it follows from the Generalized Burnside Theorem that η is an isomorphism, since $\mathrm{End}_A(M) = k$ for all the simple A-modules M.

If $\mathrm{char}(k) \neq 3$ and $\omega \notin k$, then the simple A-modules are given by $\mathsf{M} = \{M, N\}$, where $M = M_0 = A/(x-1)$ is 1-dimensional, and $N = A/(x^2+x+1) \cong k(\omega) = K$ is 2-dimensional. In this case, we have that $\mathrm{End}_A(M) = k$ and $\mathrm{End}_A(N) = K$, and we find that the standard form of A is given by

$$H = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \Rightarrow \quad A \cong \mathrm{im}(\eta) = \begin{pmatrix} k & 0 \\ 0 & K \end{pmatrix} \subseteq \mathcal{O}(\mathsf{M}) = \begin{pmatrix} k & 0 \\ 0 & \mathrm{End}_k(K) \end{pmatrix}.$$

It follows from Proposition 4 that $\eta : A \to \mathcal{O}(\mathsf{M})$ is injective. However, it is not an isomorphism in this case.

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