# A four-dimensional time-space trend model to be applied in the reconstruction of former sea levels and ice-front recession

K. KJENSTAD

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The model is represented by a four-dimensional trend function giving the elevation of the sea level (Z) above the present one as a function of locality co-ordinates (*x* and *y*) and time co-ordinate (*t*). The function and an estimate of its uncertainty have been given compact mathematical formulation by means of matrix representation. This paper only presents the model it does not apply it. No comparisons with other models are made. The model has previously been tested on data from the Oslofjord area, where it has been applied to reconstruction of the ice-front recession chronology.

*Kjell Kjenstad, Sysdeco GIS A.S., PO Box 433, N-3601-Kongsberg, Norway.*

Dating ice-front recession by means of late-glacial shore-line diagrams is a classical method in Quartenary geology. In Norwegian fjord areas this was the most common method of dating ice-front recession chronology before the introduction of the radiocarbon dating method. The shoreline diagram has been used to correlate characteristic events in sea level and ice-front recession chronology on a relative time scale. The major inconveniences of this method are the relative time scale, a coarse time resolution and projection problems in the construction of the diagram.

These difficulties were greatly reduced by the introduction of the radiocarbon dating method, and since that time knowledge about late-glacial sea level and ice-front chronologies has increased significantly. One of the major present problems consists of finding efficient methods of handling an increasing number of field observations. Furthermore, there is a need of consistent statistical methods for handling radiocarbon dating uncertainties in a time-space context.

In this text a simple numerical time-space model describing sea level changes in areas having shore-displacement curves with a single direction of curvature is presented. The model is calibrated by means of field information. These data consist of observations of former sea levels, so the principles of the method are thus classical. However, this text presents another way of handling such information. A model can be helpful in finding regularities, trends and limitations in a complex set of field information, but it is important to realise that its quality can never be better than that of the field data used to calibrate it. Most of this text is an English transcription of the theoretical aspects in part III of the author's doctor's degree thesis (Kjenstad 1984). The rest of that paper is a description of the application of the model on sea-level observations from the Oslofjord area. The quality of the model is discussed comparing a calculated ice-front recession chronology with results of independent research projects. The present paper therefore does not present any application of the model. There is therefore no sense to compare the quality of this model with similar models. This has to be left out for future papers.

The mathematical formulation of the model has also been simplified compared to Kjenstad (1984) in the sense that methods for estimation of the parameters in the model have been omitted. These are straightforward statistical methods that could be found in any textbooks of statistics or in Kjenstad (1984). The same is true for the step-by-step building up of the matrix formulas.

## The sea-level model

The dating of events connected with a certain sea level requires knowledge of the elevation of that sea level above the present one for any appropriate point of time anywhere within the research area. In mathematical language this means that the elevation of the sea-level has to be given uniquely as a function of time and space coordinates ($z = f(x, y, t)$). A convenient model for this purpose is a polynomial trend function consisting of a limited sum of terms each composed of a parameter multiplied by an increasing power of one or more of the variables. This function is chosen for its mathematical simplicity when formulated as a matrix product, and for the straightforward statistical handling of the uncertainties of the model.

When the model is applied to a certain area, the parameters of the function have to be estimated on the basis of known functional relationships between *x*, *y*, *z* and *t*. These are given by dated observations of former sea levels. These observations can be radiocarbon dates of isolation contacts in lake basins or morphological shorelines. The number of terms in the function determines the stiffness of the trend and thereby the approximation to each of the observations. In other words, this model considers each of the observations as an approximation of the respective true points in the four-dimensional vector space, and it calculates the trend in time and space between the four- dimensional observation points. It is also possible to make a certain extrapolation outside the area limited by the observation points. The general theory of the mathematical handling of trend models can be found for instance in Draper & Smith (1981).

The usual procedure when working on trend functions of three variables is to consider all three variables as having equal properties. In this particular case, however, the time variable and the space variables have different properties. Another reason for separating the variables into two categories is that the actual field information consists of information collected for the purpose of constructing either shore-displacement curves (*z* as a function of *t* in a particular locality) or isobase maps (*z* as a function of *x* and *y* at a particular point of time). Separation of the two types of variables also makes mathematical formulation of the model and estimation of the parameters easier. The formulation of the model thus starts with separate estimation of the approximation function of each shore-displacement curve to be used (hereafter the "basic curves") and a synchronous sea-level surface (hereafter the "basic sea-level"). The process is completed by combining these functions in an integrated sea-level model.

## Reconstruction of a synchronous sea-level surface ('the basic sea level') from morphological shorelines

A synchronous sea-level surface can be described analytically as a trend surface approximation to a set of morphological shorelines from an actual point in time. Speaking in terms of the sea-level model any point of time can be chosen, but it is wise to choose a point of time as far back as possible. In Sollid & Kjenstad (1980) the 'Main surface', defined as the morphological sea level of the Younger Dryas chronozone, was used as the basic sea level.

A trend surface can be expressed mathematically as:

$$Z'' = G_1'' + G_2''X + G_3''Y + G_4''X^2 + \cdots + G_k''Y^m \qquad (1)$$

The sum of terms ends when the surface has acquired a suitable stiffness. The highest order of trend-surface approximation to be solved in practice is about 6. The order of a trend-surface polynomial is defined as the highest power value in the polynomial. This means that only relatively stiff surfaces can be expressed in this way.

This function can be expressed by means of matrix formulation as the product of two matrices. One of these (**G"**) contains the estimated parameters, the other (**S"**) the different expressions of the variables *x* and *y*.

$$Z'' = \mathbf{S}''\mathbf{G}'' \qquad (2)$$

The method of estimating the parameter (coefficient) matrix **G"** is given Kjenstad (1984).

## Reconstruction of one shore-displacement curve ('one basic curve') from radiocarbon dates of isolation contacts in lake basins

Each shore-displacement curve is approximated to a polynomial of the time variable *t*. In order to get all the basic curves within the area as equal as possible, all the basic elevation values have to undergo a preliminary transformation:

$$Z' = Z / Z'' \qquad (3)$$

in areas where the total shore-displacement is rather limited, or:

$$Z' = \ln(Z+1)/\ln(Z''+1) \qquad (4)$$

in areas where strong isostatic readjustment has produced more pronounced shore-displacement and curves with a distinctly exponential form.

Both transformations hopefully result in what will be called a 'normalised' shore-displacement curve, which means that the elevation values vary between $Z' = 0$ for $t = 0$ and $Z' = 1$ for $t = t''$, where $t''$ is the age and $Z''$ the elevation of the basic sea level in the actual locality. These normalised shore-displacement curves will thus have approximately similar shape and similar elevation values, and are then approximated to the following polynomials:

$$Z'_j = D'_{1j}t + D'_{2j}t^2 + \cdots + D'_{pj}t^p \qquad (5)$$

The number of terms determines the order of the polynomial. By increasing the latter, the approximated curve can theoretically be fitted to all the data points, but the curve may then assume an improbable shape in the interval between the data points. In practice, the curve needs to be smoothed to take into account the uncertainties of the data points. This is achieved by allowing the order of the polynomial to be inferior to the number of data points. There will also be numerical problems when approximating high-order polynomials. It is therefore wise to use a low-order polynomial approximation. This is why the model in this text is only applied in areas having shore-displacement curves with a single direction of curvature, which are the only curves to be approximated to low-order polynomials. In practice, the lowest possible order of polynomials in the interval 2 to 6 are used to obtain a normalised curve with suitable stiffness, which means that it is similar to the original curve.

Curves with more than one direction of curvature can be approximated to a series of segments of low-order polynomials with continuos and smooth transitions between segments. This is a so-called 'smooth-spline' technique (deBoor 1978), and its use in sea-level problems is explained by Kjenstad (1984, part IV).

The approximation polynomial can be expressed by means of matrix formulation as the product of two matrices. One of these ($\mathbf{D}_j$) contains the estimated parameters for the normalised shore-displacement curve no. $j$, and the other (**T**) contains different expressions of the time variable $t$:

$$Z'_j = \mathbf{TD}'_j \qquad (6)$$

The method of estimating the parameter (coefficient) matrix $\mathbf{D}_j$ is given by Kjenstad (1984).

# Formulation of an integrated sea-level model

The integrated sea-level model expresses a smooth spatial transition between the normalised basic shore-displacement curves. This implies that a normalised shore-displacement curve is supposed to change its shape gradually when moving in space from the location of one basic curve to another. When extrapolating in space, the trend is assumed to continue. A spatial change in the shape of a curve with such properties can be simulated by allowing the coefficients $\mathbf{D}_i$ of a normalised shore-displacement curve be low-order polynomials (trend surfaces) of the variables $X$ and $Y$. In this way a normalised shore-displacement curve from any place can be written as:

$$Z' = D'_1(X,Y)t + D'(X,Y)_2t^2 + \cdots + D'_p(X,Y)t^p \qquad (7)$$

The coefficients $\mathbf{D}_i$ are estimated from corresponding coefficient of all the basic shore-displacement curves. This function can be expressed by means of matrix formulation as the product of three matrices.

$$Z' = \mathbf{TD'W} \qquad (8)$$

The matrix **T** contains different expressions of the time variable $t$, **D** contains the estimated coefficients of the basic shore-displacement curves, and **W** is a weighting matrix containing different expressions concerning the location of the present and the basic shore-displacement curves. The method of estimating the weighting matrix **W** is given by Kjenstad (1984).

To end up with real shore-displacement curves, the elevation values in the normalised curves have to be re-transformed inversely to the initial transformation (formulas (3) or (4) respectively).

$$Z = Z''Z' \qquad (9)$$

$$Z = \exp Z' \ln(Z''+1) - 1 \qquad (10)$$

where Z' and Z" can be substituted by their matrix formulas (8) and (2) respectively. The formulas (9) and (10) are the mathematical expressions of the integrated sea-level model, and the elevation of the sea level can thus be calculated for any point in time and for any locality. The formulas are valid within or near the four- dimensional area bounded by the data points in the basic curves and the basic sea-level. The quality of the model can be estimated by calculating the statistical variance value (see next chapter)

The model presented by Sollid & Kjenstad (1980) is based on the same principles as the present model, but it is far less general. The Sollid/ Kjenstad model is based on a single shore-displacement curve approximated to a 1. order logarithmic polynomial and it can thus be considered as a special case compared to the present model.

## An estimate of the uncertainty of the model

The reliability of the presented model, given *x*, *y* and *t*, can be estimated by calculating the statistical variance. This value (or the equivalent standard deviation value) is also a function of the variables *x*, *y* and *t*, and generally speaking, it depends on the number of basic observations, their relative location in time and space, and the uncertainty value of each of them. The general theory of statistical variance can be found in textbooks on elementary statistics, while the theory of statistical variance in regression analysis is treated by Draper & Smith (1981).

Statistically speaking, the value of the uncertainty in models based on many observations will be less than the uncertainty of each observation. This means that trend models will describe nature with less uncertainty than single observations. This is because 'noise' and stochastic deviations will be smoothed out.

Geometrically speaking, the relative four-dimensional time-space location of the basic observations will have a great influence on the uncertainty in the model. Within the four-dimensional space bounded by the basic observations, the uncertainty will remain relatively stable, while outside this area it will increase according to the degree of extrapolation. It is therefore wise to spread the basic observations in the (*x*, *y*, *t*)- space.

The total uncertainty will in general depend on the uncertainty of each observation. It is therefore necessary to make certain assumptions about the statistical distribution of each observation. These assumptions will be described later. Calculation of the statistical variance will further depend on the assumption that the order of the polynomial corresponds to the stiffness in the real processes of shore displacement. Fluctuations that are smaller than the resolution of the time and space axes are considered as "noise" with no influence on the general trend.

## An estimate of the uncertainty of a synchronous sea-level surface ('the basic sea-level')

While estimating the uncertainty of the basic sea-level surface in the model, it is assumed that all the measured elevation values of the localities indicating this surface have a Gauss distribution, with the trend-surface value as expectancy value and with the same variance. This is a reasonable assumption while working with a homogeneous set of basic observations. The calculated Z" will then also have a Gauss distribution with a variance depending on *x* and *y* and with the following matrix formulation (Draper & Smith 1981):

$$Var(Z'') = \mathbf{S''}\mathbf{Cov}(\mathbf{G''})\mathbf{S''}^{t} \qquad (11)$$

The method of estimating the covariance matrix **Cov**(**G"**) is given by Kjenstad (1984).

## An estimate of the uncertainty of one normalised shore- displacement curve ('a basic curve')

The observations fixing the basic curves are uncertain both with respect to the elevation value *z* (measurements, type of locality etc.) and in the time value *t* (radiocarbon dating etc.). Transforming to normalised elevation values involves the addition of the basic sea-level elevation uncertainty.

In trend models, uncertainty in the time value can be considered as being a part of the total elevation uncertainty, by multiplying the value of the time uncertainty by the supposed gradient of the normalised shore-displacement curve. The dating uncertainties of late-glacial and post-glacial isolation contacts have rather similar values, and normalised shore- displacement curves are assumed to be fairly straight with almost constant gradient. A relatively constant value of the time uncertainty can therefore be considered as a relatively constant factor of the normalised total elevation uncertainty.

The uncertainty of elevation measurement of the basic observations is assumed to be fairly constant from one locality to the next, which implies that all observed elevations fit a Gauss distribution with almost the same variance. Transforming to normalised elevation values involves the addition of the basic sea-level elevation uncertainty. Because both factors have Gauss distributions with a constant variance, the normalised elevation value will also have a Gauss distribution with a constant variance. The theoretical total normalised elevation uncertainty of all the localities, assuming exact indication of the time variable, consists of the elevation measurement uncertainty component, the basic sea-level surface uncertainty component and the transformed time uncertainty component. This total uncertainty will therefore also have a Gauss distribution with the same variance. The polynomial approximating to the normalised shore- displacement curve no. *j* will then have a Gauss distribution with a variance depending on *t*, and with the following matrix formulation (Draper & Smith 1981):

$$Var(Z'_j) = \mathbf{TCov}(\mathbf{D}'_j)\mathbf{T}^t \qquad (12)$$

The method of estimating the covariance matrix $\mathbf{Cov}(\mathbf{D}_j)$ is given by Kjenstad (1984).

## An estimate of the uncertainty of an integrated sea-level model

A smooth spatial transition between the basic curves is shown to be a linear combination of independent normalised elevation values $Z'_j$ with a Gauss distribution for given points of time. The weight of each $Z'_j$-value in the linear combination depends on the geometrical location in the (*x*, *y*)-plane relative to the location of the basic curves. The normalised elevation value *Z'* for a given point in time will thus have a Gauss distribution with a value of variance represented by the matrix product:

$$Var(Z') = \mathbf{W}^t\mathbf{Diag}(Var(Z'_1) \quad Var(Z'_2) \quad \cdots \quad Var(Z'_r))\mathbf{W} \qquad (13)$$

Two of the matrices are the previously defined weighting matrix **W** and the third is a diagonal matrix containing the value of the variance for each of the basic curves. When the sea-level model is estimated on the basis of one or two basic curves, the weighting matrix has a slightly different form. These special cases are treated specially by Kjenstad (1984).

When normalised elevation values are transformed back to real elevation values, the simple Gauss distributing no longer exists, and the real elevation values form a new complicated distribution. The value of the variance of these elevation values can nevertheless easily be calculated. In the linear case this can be done exactly:

$$Var(Z) = Z'^2Var(Z'') + Z''^2Var(Z') + Var(Z'')Var(Z') \qquad (14)$$

In the logarithmic case, however, an approximation formula has to be used:

$$Var(Z) \cong Z((\ln(Z''+1))^2Var(Z') + (Z'/(Z''+1))^2Var(Z'')) \qquad (15)$$

## The application of the sea-level model

Such a sea-level model can be used in four different ways:

(1) The elevation of the sea level with estimates of uncertainty at a particular locality and for a particular point in time can be calculated directly:

$$Z = f(X_0,Y_0,t_0) \pm u(X_0,Y_0,t_0) \qquad (16)$$

(2) The shore-displacement curve with estimates of uncertainty of a particular locality can be calculated directly by keeping *x* and *y* constant:

$$Z = f(X_0,Y_0,t) \pm u(X_0,Y_0,t) \qquad (17)$$

(3) The elevation of the sea-level (i.e. isobase map) for a particular point in time can be calculated directly by keeping *t* constant:

$$Z = f(X,Y,t_0) \pm u(X,Y,t_0) \qquad (18)$$

(4) The age of former sea-levels can be calculated by 'turning' the formula in such a way that the variable *t* is isolated at the left side of the equal-sign:

$$t = g(X_0,Y_0,Z_0) \pm u(X_0,Y_0,Z_0) \qquad (19)$$

The age of morphological shore-lines can be calculated with this formula. It therefore represents a possible dating method that is particularly interesting when working with localities indicating the morphological marine limit. The dating value corresponds in general to the time of deglaciation because the marine limit in most cases represents the sea level just after deglaciation of the locality. Dating of a network of such localities enables the construction of a regional chronology of ice-recession represented graphically by an isoline map. These isolines are called isocesses, and they are assumed to represent synchronous ice-front positions. The isoline-construction algorithm must also take into account the topography of the area in order to simulate realistic ice margins.

The classical method of absolute dating of ice recession in coastal areas is by means of a shoreline relation diagram (Tanner 1930). Grønlie (1941) made a mathematical formulation of the geometric relations in this diagram based on simplified geophysical models of isostasy. This method is similar to that used by Sollid & Kjenstad (1980), even though the formulations are different. However, Norwegian shore-displacement curves have proved to be different from what is assumed when using the shoreline relation diagram, and this method is thus less applicable in Norwegian areas. Andrews (1970) uses a similar model on data from the Canadian Arctic areas. Shore-displacement curves from different localities are calculated by using different sorts of Quaternary geological information. These shore-displacement curves seem to fit a common formula similar to that presented by Sollid & Kjenstad (1980).

Mørner (1974) presents a formula which describes the relation between the gradient of the highest shore-line, the rate of the relative land uplift, the gradient of the marine limit and the rate of ice-recession. If three of these factors are known, the fourth is given explicitly. This formula can therefore be used for relative dating of ice-recession chronology by using information from ordinary equidistant shoreline diagrams.

## Conclusions

The presented sea-level reconstruction model has previously been applied to sea- level observations from the Oslofjord area, and have led to calculated results that are in acceptable agreement with independent observations from the area (Kjenstad 1984). The main advantage of such a model is its ability to integrate a complicated set of four-dimensional time-space observations in a continuous time-space description. The results can be used to discover trends and limitations in complex natural processes, and to analyse the statistical properties of the field information. Such models are necessary for the construction of the palaeogeography of an area. The models can therefore be a useful tool in Quaternary research.

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