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RAPPORT

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The Non Commutative Compactification of r-bundles

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HØGSKOLEN i Buskerud

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1 Introduction

We consider an n-dimensional variety X over an algebraically closed field k with an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. Let

$$M^{ss}_{X,\mathcal{L}}(r, c_1, \ldots, c_{\min(r,n)})$$

be the moduli space of rank r, semistable locally free $\mathcal{O}_X$-modules with chern classes

$$c_i \in H^{2i}(X, \mathbb{Z}).$$

When X is a smooth variety, This has been studied by Laura Costa and Rosa M.Miró-Roig in [1].

From geometric invariant theory, [2], we have the following:

Theorem 1. *Fix X, H, r and algebraic cycles $a_1,\ldots,a_k$, $k = \min(r, s\ l)$ up to numerical equivalence ($\operatorname{codim} a_i = i$). Then the set of semistable torsion-free sheaves E with $c_i(E)$ numerically equivalent to a_i, modulo the equivalence relation $\mathcal{E} \sim \mathcal{F}$ if $\operatorname{gr}\mathcal{E} \cong \operatorname{gr}\mathcal{F}$ is in a natural way the set of points of a scheme $\mathcal{U}^{ss}_X(r; a_1, \ldots, a_k)$ locally of finite type. If $n = 2$, $r = 2$ or $\operatorname{char}(k) = 0$, $\mathcal{U}^{ss}_X$ is projective.*

It is conjectured in [2] that $\mathcal{U}^{ss}_X(r; a_1, \ldots, a_k)$ is always projective. However, we restrict our attention in this paper to $\operatorname{char}(k) = 0$ and may therefore assume that $\mathcal{U}^{ss}_X$ is projective.

With these assumptions, we can consider this as a compactification. That is the following: Consider the diagram

$$\begin{array}{ccc} M^{ss}_{X,\mathcal{L}}(r, c_1, \ldots, c_{\min(r,n)}) & \longrightarrow & \mathcal{U}^{ss}_X(r; a_1, \ldots, a_k) \\ \downarrow & & \downarrow \\ M_{X,\mathcal{L}}(r, c_1, \ldots, c_{\min(r,n)}) & \longrightarrow & \mathcal{U}_X(r; a_1, \ldots, a_k) \end{array} .$$

Here the moduli spaces without the superscript ss denotes the corresponding not necessarily commutative moduli of all objects, i.e. not only the semistable or stable ones. We are interested in the boundary of the semistable spaces in the non commutative moduli.

We would like to consider these non commutative moduli as compactifications of the commutative ones, and as such we are interested in the local structure, see [12] for a commutative example.

In this paper, k is always an algebraically closed field of characteristic 0 and A denotes a finite type k-algebra.

2 Deformation theory for $\mathcal{O}_X$-modules

2.1 Affine deformations

Let $V = \{V_1, \ldots, V_r\}$ be right A-modules. Let $S = (S_{ij}) \in \underline{a}_r$ be an r-pointed artinian k-algebra, that is an artinian k-algebra S together with morphisms satisfying the diagram

$$\begin{array}{ccc} k^r & \longrightarrow & S \\ & \overset{\mathrm{Id}}{\searrow} & \downarrow \\ & & k^r \end{array}$$

The deformation functor

$$Def_V : \underline{a}_r \longrightarrow \underline{\text{Sets}}$$

is defined by

$$\mathrm{Def}_V(S) = \{S \otimes_k A\text{-modules } M_S | k_i \otimes_S M_S \cong V_i \text{ and } M_S \cong_k (S_{ij} \otimes_k V_j) = S \otimes_k V\} / \cong .$$

Notice that the condition S-flat in the commutative case is replaced by $M_S \cong_k (S_{ij} \otimes_k V_j)$ in the non commutative case (here $\cong_k$ means isomorphic as k-vectorspaces).

The obstruction theory for the non commutative deformation functor is given by the following:

Let $M_S \in \mathrm{Def}_V(S)$. Then $M_S \cong_k (S_{ij} \otimes_k V_j)$ and as such it has an obvious structure as left $S = (S_{ij})$-module. The (right) A-module structure is determined by the k-algebra homomorphism

$$A \xrightarrow{\sigma} \mathrm{End}_k(M_S) \Leftrightarrow A \xrightarrow{\sigma} \mathrm{End}_k(S_{ij} \otimes_k V_j).$$

We let $k^r = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix}$ and by k_i we understand $e_i \cdot k^r$.

Now, $\sigma(a) : A \longrightarrow \mathrm{End}_k(S_{ij} \otimes_k V_j)$ is induced by

$$\sigma(a) : (k^r \otimes_k V_j) = \begin{pmatrix} V_1 \\ \vdots \\ V_r \end{pmatrix} \longrightarrow (S_{ij} \otimes_k V_j).$$

Let $v_k \in V_k$. The linearity of $\sigma(a)$ over S gives that $\sigma(a)(v_k) = \sigma(a)(e_k \cdot v_k) = e_k\sigma(a)(v_k) \in \begin{pmatrix} 0 & \cdots & 0 \\ S_{k,1} \otimes_k V_1 & \cdots & S_{k,r} \otimes_k V_r \\ 0 & \cdots & 0 \end{pmatrix}$. Thus $\sigma(a)$ is completely determined by the morphisms

$$\sigma_{ij}(a) : V_i \longrightarrow S_{ij} \otimes_k V_j.$$

Going the other way, any k-algebra homomorphism induced by such σ_{ij}'s commuting in the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\sigma_{ij}(a)} & S_{ij} \otimes_k V_j \\ & \searrow^{\tilde{\sigma}_i(a)} & \uparrow \\ & & V_i \end{array}$$

where $\tilde{\sigma}_i(a)$ is the given right A-module structure of V_i, defines a deformation of V to S.

Let M_S be the deformation of V to S given by the k-algebra homomorphism $\sigma^S : A \longrightarrow \mathrm{End}_k(S_{ij} \otimes V_j)$ inducing as above

$$V_i \xrightarrow{\sigma^S_{ij}(a)} S_{ij} \otimes_k V_j.$$

Let

$$(R_{ij}) = R \xrightarrow{\pi} S = (S_{ij})$$

be a small morphism, i.e. $\ker \pi \cdot \mathrm{rad}(R) = (I_{ij}) \cdot \mathrm{rad}(R) = 0$. We may lift $\sigma_{ij}(a)$ in the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\sigma^R_{ij}(a)} & R_{ij} \otimes_k V_j \\ = \downarrow & & \downarrow \\ V_i & \xrightarrow{\sigma^S_{ij}(a)} & S_{ij} \otimes_k V_j \end{array}$$

by adding to σ^S_{ij} any k-linear morphism $\theta_{ij} : A \longrightarrow \mathrm{Hom}_k(V_i, I_{ij} \otimes_k V_j)$. The obvious lifting is of course the trivial one, i.e. choosing $\theta = 0$. Choosing the lifting σ^R this way, the morphism

$$A \longrightarrow \mathrm{End}_k(R_{ij} \otimes_k V_j)$$

is k-linear, and the one thing left for this to be an A-module structure on $R \otimes_k V$ is the condition $\sigma^R(ab) = \sigma^R(a)\sigma^R(b)$. Because this condition holds for S, we get an element

$$\psi^R(a,b) = \sigma^R(ab) - \sigma^R(a)\sigma^R(b) : V_i \longrightarrow R_{ij} \otimes_k V_j$$

commuting in the diagram

$$\begin{array}{ccc} & & I_{ij} \otimes_k V_j \\ & \nearrow & \downarrow \\ V_i & \xrightarrow{\psi^R(a,b)} & R_{ij} \otimes_k V_j \\ & \searrow^{0} & \downarrow \\ & & S_{ij} \otimes_k V_j \end{array}.$$

Thus we actually have $\psi^R(a,b) : V_i \longrightarrow I_{ij} \otimes_k V_j$. Because π is a small morphism, we have $I^2 = 0$ and thus $a \cdot \psi^R(b,c) = \sigma^R(a)\psi^R(b,c)$ and $\psi^R(a,b)\cdot c = \psi^R(a,b)\sigma(c)$.

Letting d be the Hochschild coboundary map, we now find

$$\begin{aligned}
d(\psi^R)(a,b,c) &= a\psi^R(b,c) - \psi^R(ab,c) + \psi^R(a,bc) - \psi^R(a,b)c \\
&= a(\sigma^R(bc) - \sigma^R(b)\sigma^R(c)) \\
&- (\sigma^R(abc) - \sigma^R(ab)\sigma^R(c)) \\
&+ \sigma^R(abc) - \sigma^R(a)\sigma^R(bc) \\
&- (\sigma^R(ab) - \sigma^R(a)\sigma^R(b))c \\
&= a\sigma^R(bc) - \sigma^R(a)\sigma^R(bc) \\
&+ \sigma^R(a)\sigma^R(b)c - a\sigma^R(b)\sigma^R(c) \\
&+ \sigma^R(ab)\sigma^R(c) - \sigma^R(ab)c = 0
\end{aligned}$$

Definition 1. *Given a small morhism $\pi : R \longrightarrow S$ between r-pointed artinian k-algebras and $M_S \in \mathrm{Def}_V(S)$, we define the obstruction $o(\pi, M_S) = (o_{ij}(\pi, M_S))$ for lifting M_S to R as the class of*

$$\psi_{ij}^R : A^{\otimes 2} \longrightarrow (I_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$$

in $\mathrm{HH}^2(A, \mathrm{Hom}_k(V_i, V_j))$.

Theorem 2. $o(\pi, M_S) = 0$ *if and only if there exists a lifting $M_R \in \mathrm{Def}_V(R)$ of M_S. The set of isomorphism classes of such liftings is a torsor under*

$$(I_{ij} \otimes_k \mathrm{Ext}_A^1(V_i, V_j)).$$

Proof. The complete proof can be found in [5]. We will just state what is essential for this work:

Assume $0 = o_{ij} = \bar{\psi_{ij}} \in \mathrm{HH}^2(A, \mathrm{Hom}_k(V_i, V_j))$. Then $\psi = d\phi$, $\phi \in \mathrm{Hom}_k(A, I_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$. Put $\sigma' = \sigma + \phi$. Then

$$\begin{aligned}
\sigma'(ab) - \sigma'(a)\sigma'(b) &= \sigma(ab) + \phi(ab) - (\sigma(a) + \phi(a))(\sigma(b) + \phi(b)) \\
&= \sigma(ab) - \sigma(a)\sigma(b) + \phi(ab) - \sigma(a)\phi(b) - \phi(a)\sigma(b) - \phi(a)\phi(b) \\
&= \sigma(ab) - \sigma(a)\sigma(b) - (\sigma(a)\phi(b) - \phi(ab) + \phi(a)\sigma(b)) - \phi(a)\phi(b) \\
&= \psi - d\phi - 0 = 0.
\end{aligned}$$

This is because as before $I^2 = 0 \Rightarrow \phi^2 = 0$, $a \cdot \phi(b,c) = \sigma(a)\phi(b,c)$ and $\phi(a,b) \cdot c = \phi(a,b) \cdot \sigma(c)$. □

2.2 Non commutative obstruction theory in the Yoneda complex

Even in the commutative case, we are missing a suitable reference. This non commutative theory then includes the commutative case, and is essential for the development of Massey products in the category of $\mathcal{O}_X$-modules. Let $\{V_1, ..., V_r\}$ be right A-modules. Choose free (projective) resolutions $L_.^i$.

Theorem 3. *Let $\phi : R \longrightarrow S$ be a small morphism in the category of r-pointed artinian k-algebras. Then $V_S \in \mathrm{Def}_V(S)$ can be lifted to $V_R \in \mathrm{Def}_V(R)$ if and only if there exists a lifting of complexes*

$$(R_{ij} \otimes_k L^j_\cdot) \longrightarrow (S_{ij} \otimes_k L^j_\cdot).$$

Proof. The proof goes in various steps: First, we have to prove that every $V_S \in \mathrm{Def}_V(S)$ has a resolution of the form

$$V_S \longleftarrow (S_{ij} \otimes_k L^j_\cdot)$$

This is obviously true for $S = k^r = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix}$:

$$0 \longleftarrow \begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_r \end{pmatrix} \longleftarrow \begin{pmatrix} L^1_\cdot & & 0 \\ & \ddots & \\ 0 & & L^r_\cdot \end{pmatrix},$$

so because $S \in \mathrm{ob}(\underline{a}_r)$, it is enough to prove that if $V_S \in \mathrm{Def}_V(S)$ has a resolution, then V_S can be lifted to $V_R \in \mathrm{Def}_V(R)$ if and only if the resolution of V_S can be lifted to R. Consider the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & (I_{ij} \otimes_k V_j) & \xleftarrow{\varepsilon_{\mathrm{Id}}} & (I_{ij} \otimes_k L^j_0) & \xleftarrow{d^{\mathrm{Id}}_0} & (I_{ij} \otimes_k L^j_1) & \xleftarrow{d^{\mathrm{Id}}_1} & (I_{ij} \otimes_k L^j_2) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l_0} & & \downarrow & & \downarrow & & \\
 & & M_R & \xleftarrow[\varepsilon_R]{} & (R_{ij} \otimes_k L^j_0) & \xleftarrow[d^R_0]{} & (R_{ij} \otimes_k L^j_1) & \xleftarrow[d^R_1]{} & (R_{ij} \otimes_k L^j_2) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_0} & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & M_S & \xleftarrow[\varepsilon_S]{} & (S_{ij} \otimes_k L^j_0) & \xleftarrow[d^S_0]{} & (S_{ij} \otimes_k L^j_1) & \xleftarrow[d^S_1]{} & (S_{ij} \otimes_k L^j_2) & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Assume first that a lifting M_R exists. Then by definition $M_R \cong_k (R_{ij} \otimes_k I_{ij})$ and thus the first vertical column is exact. For the same reason, the top row is exact. Then because L^j_0 is projective, we can lift ε_S to ε_R. To continue the process of lifting the bottom row to the middle, we must prove that $\ker \varepsilon_R$ maps surjectively to $\ker \varepsilon_S$. The surjectivity is proved the following way: Let $\bar{x} = \rho_0(x) \in \ker(\varepsilon_S)$. Then $\rho(\varepsilon_R(x)) = 0 \Rightarrow \varepsilon_R(x) = l(y')$ where $y' = \varepsilon_{\mathrm{Id}}(y)$ for some $y \in (I_{ij} \otimes_k L^j_0)$. Then $\rho_0(x - l_0(y)) = \bar{x}$ and $\varepsilon_R(x - l_0(y)) = \varepsilon_R(x) -$

$\varepsilon_R(l_0(y)) = \varepsilon_R(x) - l(\varepsilon_I(y)) = \varepsilon_R(x) - l(y') = 0$. Given this surjectivity we can lift d_0^S to d_0^R. Continuing this process on the kernel, we can lift to a sequence of morphisms $d_.^R$ such that $(d_.^R)^2 = 0$. We have not yet proved that the sequence above is exact, but this follows from the converse argument: Assume conversely that such a lifting as above exists, i.e. such that $(d_.^R)^2 = 0$. The long exact sequence gives

$$\cdots \longrightarrow H^i(I \otimes_k L_.) \longrightarrow H^i(R \otimes_k L_.) \longrightarrow H^i(S \otimes_k L_.) \longrightarrow \cdots \longrightarrow$$

$$H^1(I \otimes_k L_.) \longrightarrow H^1(R \otimes_k L_.) \longrightarrow H^1(S \otimes_k L_.) \longrightarrow$$

$$H^0(I \otimes_k L_.) \longrightarrow H^0(R \otimes_k L_.) \longrightarrow H^0(S \otimes_k L_.) \longrightarrow 0$$

The exactness of the top and bottom row then implies the exactness of the middle row and the first column. In particular

$$M_R \cong H^0(R \otimes_k L_.) \cong_k (R_{ij} \otimes_k V_j).$$

□

See [13] for an extensive treatment of the affine obstruction theory in the Yoneda complex for the non commutative case.

2.3 The spectral sequence of $\mathrm{Ext}_X(\mathcal{F}, \mathcal{G})$

Let X be a n-dimensional scheme, $\mathcal{F}$, $\mathcal{G}$ two $\mathcal{O}_X$-modules. Then $\mathrm{Ext}^i_X(\mathcal{F}, \cdot) = R^i \mathrm{Hom}_X(\mathcal{F}, \cdot)$ and $\mathcal{E}xt^i_X(\mathcal{F}, \cdot) = R^i \mathcal{H}om_X(\mathcal{F}, \cdot)$. The category of $\mathcal{O}_X$-modules has enough injectives such that these homology groups are well defined. Now these groups are important for computational aspects of our theory, and in general for all moduli theory. The injective modules are not very well suited for computations. On the other hand, projective modules, and in particular free modules are very well suited for our computations. It is well known that the category of $\mathcal{O}_X$-modules has not enough projectives, but restricting to quasi coherent $\mathcal{O}_X$-modules we can solve the problem partially.

Lemma 1. *Let $\mathcal{U}$ be an open cover of a scheme X. Assume that for each open $U \in \mathcal{U}$ we have given an $\mathcal{O}_U$-module $\mathcal{F}^U$, for each couple $U, V \in \mathcal{U}$, an isomorphism $\phi_{UV} : \mathcal{F}^U|_{U \bigcap V} \cong \mathcal{F}^V|_{U \bigcap V}$. such that $\phi_{UU} = \mathrm{Id}$, $\phi_{UV} \circ \phi_{VW} = \phi_{UW}$ on $U \bigcap V \bigcap W$. Then the gluing of the family $\{\mathcal{F}^U\}_{U \in \mathcal{U}}$ is*

$$H^0(\check{\mathcal{C}}(\mathcal{U}, \{\mathcal{F}^U\}_{U \in \mathcal{U}})).$$

Proof. This is just the fact that $H^0(\check{\mathcal{C}}(\mathcal{U}, \{\mathcal{F}^U\}_{U \in \mathcal{U}}))$ is a sheaf on X. □

Lemma 2. *Every coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a quasi compact scheme X is a quotient of a locally free $\mathcal{O}_X$-module.*

Proof. The category of A-modules has enough locally free's, so for an open affine cover $\mathcal{U} = \{U_i = \text{Spec}(A_i)\}$ of X we can choose surjections

$$\mathcal{L}^{U_i} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0$$

of $\mathcal{O}_{U_i}$-modules where $\mathcal{L}^{U_i}$ is locally free (projective) and compatible on the intersections. Then the induced morphism

$$\mathcal{L} = H^0(\check{\mathcal{C}}(\mathcal{U}, \{\mathcal{L}^U\}_{U\in\mathcal{U}})) \longrightarrow H^0(\check{\mathcal{C}}(\mathcal{U}, \{\mathcal{F}|_U\}_{U\in\mathcal{U}})) = \mathcal{F}$$

is surjective. □

Lemma 3. *Let* $0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{L}_.$ *be a locally free resolution of* $\mathcal{F}$. *Then* $\mathcal{E}\text{xt}^i_X(\mathcal{F}, \mathcal{G}) = H^i(X, \mathcal{H}\text{om}(\mathcal{L}_., \mathcal{G}))$.

Proof. Hartshorne[4] □

Theorem 4. *(Godement) There is a spectral sequence with*

$$E_2^{p,q} \cong H^p(H^q(X, \mathcal{E}\text{xt}^p(\mathcal{F}, \mathcal{G}))$$

such that $E_2^{p,q} \Rightarrow \text{Ext}^n_X(\mathcal{F}, \mathcal{G})$.

Proof. All the details of this proof can be found in Godement[10].Let C^{**} be the double complex $C^*(\mathcal{U}, \mathcal{H}\text{om}(\mathcal{F}, \mathcal{I}^*))$ where $\mathcal{I}^*$ is an injective resolution of $\mathcal{G}$. Then ${}'E_2^{p,q} =' H^p(''H^q(C^{**})) =' H^p(\mathcal{U}, \mathcal{E}\text{xt}^p(\mathcal{F}, \mathcal{G})) = H^p(X, \mathcal{E}\text{xt}^p(\mathcal{F}, \mathcal{G}))$ and ${}''E_2^{p,q} = {}''H^p('H^q(C^{**})) = H^p(H^q(X, \mathcal{H}\text{om}(\mathcal{F}, \mathcal{I}^*))$. Thus
${}''E_2^{n,0} = H^n(H^0(X, \mathcal{H}\text{om}(\mathcal{F}, \mathcal{I}^*))) = H^n(\text{Hom}(\mathcal{F}, \mathcal{I}^*)) = \text{Ext}^n_X(\mathcal{F}, \mathcal{G})$. □

Definition 2. *Let* X/k *be a separated, noetherian scheme,* $\mathcal{F}$ *a quasi coherent* $\mathcal{O}_X$*-module. We will say that* $\mathcal{F}$ *has support strictly inside an open affine subset* $U_0 = \text{Spec}(A)$ *if there exists an open affine covering* $\mathcal{U} = \{U_i\}_{i\in I}$ *of* X *containing* U_0 *such that* $\mathcal{F}(U_0 \cap U_p) = 0$ *when* $p \neq 0$

Lemma 4. *Let* X/k *be a separated noetherian scheme,* $\mathcal{F}$, $\mathcal{G}$ *two quasi-coherent* $\mathcal{O}_X$*-modules. Assume that* $\mathcal{E}\text{xt}^i_X(\mathcal{F}, \mathcal{G})$ *has support strictly inside an open affine subset* $U_0 = \text{Spec}(A)$. *Then for any open* $U \subseteq X$ *containing* U_0*, we have that*

$$\text{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong \text{Ext}^i_U(\mathcal{F}, \mathcal{G}).$$

Proof. Using Godement's spectral sequence [10] on the Check-complex, we find that

$$\text{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{p+q=i} E_\infty^{p,q}$$

with $E_2^{p,q} = H^p(X. \mathcal{E}\text{xt}^q_X(\mathcal{F}, \mathcal{G}))$. Again, the Check complex applied to $\mathcal{E}\text{xt}^q_X(\mathcal{F}, \mathcal{G})$, gives $H^p(X, \mathcal{E}\text{xt}^q_X(\mathcal{F}, \mathcal{G})) = 0$ when $p > 0$. This leaves us with $\text{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong E_\infty^{0,i}$. As $E_2^{0,i} \cong H^0(X; \mathcal{E}\text{xt}^i_X(\mathcal{F}, \mathcal{G})) \cong H^0(U; \mathcal{E}\text{xt}^i_U(\mathcal{F}|_U, \mathcal{G}|_U))$ This finally gives

$$\text{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong \text{Ext}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$$

□

2.4 Global Deformation Theory

Here we recall the basic notions of global obstruction theory, that is the theory of deformations of sheaves of $\mathcal{O}_X$-modules. The theory works in a much more general setting, but here we will assume that X is a separated, noetherian scheme, and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Notice that $\mathrm{Def}_{\mathcal{F}} : \underline{l} \longrightarrow \underline{\mathrm{Sets}}$ is given by

$$\mathrm{Def}_{\mathcal{F}}(S) = \{\mathcal{O}_{X\times_k S} - \text{modules } \mathcal{F}_S | \mathcal{F}_S \otimes_S k(*) \cong \mathcal{F}, \mathcal{F}_S \text{ is } S\text{-flat}/\} \sim .$$

Lemma 5. *Assume that* $0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$ *is a small morphism in* $\underline{l}$*, and let* $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$*. Then*

$$\{\textit{Liftings } \mathcal{F}_R \textit{ of } \mathcal{F}_S \textit{ to } R\} \cong \{0 \longrightarrow I \otimes_k \mathcal{F} \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0\}/\sim .$$

Proof. If $\mathcal{F}_R$ is a lifting of $\mathcal{F}_S$ to R, then $\mathcal{F}_R$ is R-flat, i.e. $0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0 \Rightarrow 0 \longrightarrow I \otimes_R \mathcal{F}_R \longrightarrow R \otimes_R \mathcal{F}_R \longrightarrow S \otimes_R \mathcal{F}_R \longrightarrow 0 \Rightarrow 0 \longrightarrow I \otimes_R \mathcal{F}_R \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0$. This is because $I \otimes_R \mathcal{F}_R \cong I \otimes_k (k \otimes_R \mathcal{F}_R) = I \otimes_k \mathcal{F}$.

Conversely, if $0 \longrightarrow I \otimes_k \mathcal{F} \longrightarrow \mathcal{F}_R \longrightarrow \mathcal{F}_S \longrightarrow 0$ is exact, then $\mathcal{F}_R$ is R-flat and the sequence

$$0 \longrightarrow (I \otimes_k \mathcal{F}) \otimes_R S \longrightarrow \mathcal{F}_R \otimes_R S \longrightarrow \mathcal{F}_S \otimes_R S \longrightarrow 0$$

is exact. But the image of $(I \otimes_k \mathcal{F}) \otimes_R S$ in $\mathcal{F}_R \otimes_R S$ is 0, and

$$\mathcal{F}_S \otimes_R S = \mathcal{F}_S \otimes_R R/I = \mathcal{F}_S.$$

Thus $\mathcal{F}_R \otimes_R S \cong \mathcal{F}_S$. □

Corollary 1.

$$\mathrm{T}_{\mathrm{Def}_{\mathcal{F}}} \cong \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F}).$$

Given now an $\mathcal{O}_X$-module $\mathcal{F}$ on X (separated, noetherian over algebraically closed field k), Choose a locally free resolution

$$0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{L}.,$$

and choose an open affine covering $\mathcal{U} = \{U_i = \mathrm{Spec}\, A_i\}_{i\in I}$ of X such that $\mathcal{L}_p$ is free for each p. For this setup we have:

Lemma 6. *The following are equivallent*

a) To give a lifting $\mathcal{F}_S$ *of* F *to* $S \in \underline{l}$

b) To give morphisms

$$d_i : \mathcal{L}.(U_i) \otimes_k S \longrightarrow \mathcal{L}.(U_i)(-1) \otimes_k S,\ \phi_{ij} : \mathcal{L}.(U_i \cap U_j) \otimes_k S \longrightarrow \mathcal{L}.(U_i \cap U_j) \otimes_k S$$

such that $d_i^2 = 0$*,* $d_i \circ \phi_{ij} - \phi_{ij} \circ d_j = 0$*,* $\phi_{ij} \circ \phi_{jk} - \phi_{ik} = 0$*.*

c) To give a lifting of double complexes

$$\check{C}(\mathcal{U} \otimes_k S, \mathcal{L}. \otimes_k S) \longrightarrow \check{C}(\mathcal{U}, \mathcal{L}.).$$

Proof. The proof follows as in the proof of theorem 3. Then the globalization is taken care of by lemma 1 □

Proposition 1. *Let* $0 \longrightarrow R \xrightarrow{\phi} S \longrightarrow 0$ *be a small morphism in* $\underline{l}$. *Then for each* $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(S)$ *there exists an element*

$$o(\phi, \mathcal{F}_S) \in \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})$$

such that $\mathcal{F}_S$ *can be lifted to* R *if and only if* $o(\phi, \mathcal{F}_S) = 0$. *Furthermore, if this is true then* $\mathrm{Def}_{\mathcal{F}}(R)$ *is a torsor (principal homogeneous space) over* $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$.

Proof. This is done completely in [12]. We will however write up the expression for the obstruction in this case:

Consider the small morphism $0 \longrightarrow R \xrightarrow{\phi} S \longrightarrow 0$ and let $\mathcal{F}_S \in \mathrm{Def}_{\mathcal{F}}(R)$ be given by the morphisms

i)

$$d_i^S : \mathcal{L}.|_{U_i} \otimes_k S \longrightarrow \mathcal{L}.(-1)|_{U_i} \otimes_k S,$$

ii)

$$\phi_{ij}^S : \mathcal{L}.|_{U_i \cap U_j} \otimes_k S \longrightarrow \mathcal{L}.|_{U_i \cap U_j} \otimes_k S.$$

These morphisms satisfies

1)

$$(d_i^S)^2 = 0 \text{ for all } i \in I$$

2)

$$d_i^S \circ \phi_{ij}^S - \phi_{ij}^S \circ d_j^S = 0$$

3)

$$\phi_{ij}^S \circ \phi_{jk}^S - \phi_{ik}^S = 0.$$

We can lift these morphisms in the obvious (free) manner to d_i^R and ϕ_{ij}^R. Then the obstruction is given by

$$o = [(d_i^R)^2, d_i^R \circ \phi_{ij}^R - \phi_{ij}^R \circ d_j^R, \phi_{ij}^R \circ \phi_{jk}^R - \phi_{ik}^R]$$

which is an element in

$$\check{C}^0(X; \mathcal{H}\mathrm{om}^2(\mathcal{L}., \mathcal{L}.\otimes_k I)) \oplus \check{C}^1(X; \mathcal{H}\mathrm{om}^1(\mathcal{L}., \mathcal{L}.\otimes_k I)) \oplus \check{C}^2(X; \mathcal{H}\mathrm{om}^0(\mathcal{L}., \mathcal{L}.\otimes_k I)).$$

It follows that $d(o) = 0$ in the total complex $C^{\cdot}$ of

$$\check{C}^{\cdot}(X; \mathcal{H}\mathrm{om}^{\cdot}(\mathcal{L}., \mathcal{L}. \otimes_k I))$$

giving the class of the obstruction, that is

$$o \in H^2(C^{\cdot}) \cong \mathrm{Ext}^2_X(\mathcal{F}.\mathcal{F}) \otimes_k I.$$

Notice that this generalizes to the noncommutative situation exactly as in the affine situation. □

Notice that we can use any resolving functor for $\varprojlim^{(n)}$. Thus this can also be done by using the functors of Laudal[8] or Godement[10].

For computations, we need the following: Let $K^{\cdot,\cdot}$ be a double complex with differentials

$$\begin{array}{ccc} K^{p,q} & \xrightarrow{''d} & k^{p,q+1} \\ \downarrow{'d} & & \downarrow{'d} \\ K^{p+1,q} & \xrightarrow[''d]{} & k^{p+1,q+1} \end{array}$$

Then the total complex is

$$K^n = \oplus_{p+q=n} K^{p,q}$$

with differential $d^n : K^n \longrightarrow K^{n+1}$ given by $d^n = {}'d + (-1)^n {}''d$.

Also recall that the Čheck complex is given by

$$\check{C}^p(\mathcal{U};\mathcal{F}) = \prod_{i_0<i_1<\cdots<i_p} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}).$$

The differential is $d^p : \check{C}^p(\mathcal{U};\mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U};\mathcal{F})$,

$$(d^p\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_p}}.$$

3 Chern Classes and Stability

Here we would like to recall the definition of Chern classes and stable sheaves. Let X be an n-dimensional, non singular variety with a rank r-bundle $\mathcal{F}$, i.e. a locally free $\mathcal{O}_X$-module of rank r. An m-cycle on X is an irreducible variety of codimension m and the free abelian group generated by the m-cycles modulo rational equivalence is called $A^m(X)$. For a morphism

$$f : X \longrightarrow X'$$

we have the pushdown $f_* : A^m(X) \longrightarrow A^m(X')$ and the pullback $f^* : A^m(X') \longrightarrow A^m(X)$. An intersection theory on a class of abelian varieties is a pairing $A^r(X) \times A^s(X) \longrightarrow A^{r+s}(X)$ which makes

$$A(X) = \bigoplus_{r=0}^{n} A^r(X)$$

to an associative, commutative, graded ring with identity, the Chow ring, such that the obvious conditions holds (that are the ones inherited from the intersection theory of curves and surfaces). The essentials for the definition of Chern classes is given by the following:

Lemma 7. *Let* $\xi \in A^1(\mathbb{P}(\mathcal{F}))$ *be the class of the divisor corresponding to* $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ *and let* $\pi : \mathbb{P}(\mathcal{F}) \longrightarrow X$ *be the projection. Then*

$$\pi^* : A(X) \longrightarrow A(\mathbb{P}(\mathcal{F}))$$

makes $A(\mathbb{P}(\mathcal{F}))$ *to a free* $A(X)$*-module generated by* $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

Definition 3. $c_i(\mathcal{F}) \in A^i(X)$ *is given by* $c_0(\mathcal{F}) = 1$ *and*

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(s\ F).\xi^{r-i} = 0.$$

Rewritten, this is equivalent to $\xi^r = \sum_{i=1}^{r} (-1)^{i+1} \pi^* c_i(\mathcal{F}).\xi^{r-i}$.

As $c_0 = 1$, $A^i(X) = 0$ for $i > n$ and $c_i(\mathcal{F})$ is defined for $0 \leq i \leq r$, the Chern classes of a rank r bundle $\mathcal{F}$ on an n-dimensional variety is determined by $c_i(\mathcal{F})$, $0 < i \leq \min(r, n)$. Notice that the definition of Chern classes can be extended to coherent sheaves by extensions with bundles (again we make use of the fact that the category of coherent sheaves has enough locally frees).

In [2] the following definitions are given.

Definition 4. *Let* X *be a smooth curve with a bundle* $\mathcal{E}$. *Then* $\mathcal{E}$ *is stable (respectively semistable) if*

$$\deg(c_1(\mathcal{F})) < \deg(c_1(\mathcal{E})) \cdot \frac{\mathrm{rk}(\mathcal{F})}{\mathrm{rk}(\mathcal{E})}, \textit{ (respectively } \leq \textit{)}$$

for all proper sub-bundles $\mathcal{F} \subset \mathcal{E}$.

Definition 5. *Let* X *be a smooth* n*-dimensional projective variety with hyperplane section* H. *A torsionfree sheaf* $\mathcal{E}$ *on* X *is called stable (respectively semistable) if*

$$\chi(\mathcal{F}(nH)) < \frac{\mathrm{rk}(\mathcal{F})}{\mathrm{rk}(\mathcal{E})} \cdot \chi(\mathcal{E}(nH)), \textit{ for } n >> 0 \textit{ (respectively } \leq \textit{)}$$

for all proper sub-bundles $\mathcal{F}$ *of* $\mathcal{E}$.

Notice that in 4 the deg prefix is not in [2]. I have added it so that the definition makes sence and because it fits in with definition 5. Also it fits in with the definition of Hartshorne in[4] if we define the degree of a bundle $\mathcal{F}$ as $\deg(c_1(\mathcal{F}))$ and use the Hirzebruch-Riemann-Roch theorem[4].

4 The Moduli Problem for Bundles

4.1 Introduction

If a moduli space is proved to exist, local neighborhoods can be found by ordinary deformation theory. Also, in cases where we do not have to few generic points, the non commutative algebraic geometry can be used to glue the local formal moduli together to a moduli space. The non commutative algebraic geometry is essential when it comes to those classes of modules where it is proved that a moduli space does not exists because the properties of the orbits is not satisfied under any reductive group action. That is, the objects corresponds to points in a scheme X that is not stable in any form for any polarization. To study the non commutative boundary of the semistable bundles, we have to understand the construction of the schemes

$$\mathcal{U}_X^{ss}(r; a_1, ..., a_k).$$

Let X be a n-dimensional, smooth, irreducible, algebraic variety over $\mathbb{C}$, and let $\mathcal{L}$ be an ample divisor on X. Because X is an irreducible variety, X is integral, and then every locally free $\mathcal{O}_X$-module is torsionfree. Thus

$$\mathcal{M}_{X,\mathcal{L}}^s \subseteq \mathcal{M}_{X,\mathcal{L}}^{ss} \subseteq \mathcal{U}_X^{ss},$$

and the general construction given or referred to in GIT[2] holds.

Example 1. *Consider a smooth projective curve X. Then we know the existence of the Jacobian variety* Jac(X). *Then*

$$\mathcal{M}_{X,\mathcal{L}}^s \subseteq \mathcal{M}_{X,\mathcal{L}}^{ss} \subseteq \mathrm{Jac}(X)$$

for a suitable group-action, and the problem can be solved by commutative methods.

Remark 1. *Theorem 1.10 in GIT[2] states that*

$$X^{ss}/G \supseteq X_{(0)}^s/G$$

exists. The names (semi-) stable for bundles does not a priori mean that the bundles are (semi-) stable for some reductive group-action. We are not supposed to take the quotient, rather representing these bundles as the points in a quotient.

In the following we will try to work as general as possible. However, when we need to, we restrict to the following case: $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}_X(1)$ and we will investigate $\mathcal{M}_{X,\mathcal{L}}^{ss}(2, c_1, c_2)$, the moduli space of rank 2 vector bundles on X with Chern classes numerically equivalent to c_1 and c_2.

4.2 $A - G$ Modules

Let A be a k-algebra, G a group, M a (right) A-module. Assume that G acts (dually) on A and M by

$$\nabla : A \longrightarrow \mathrm{Aut}_k(A),\ \nabla : A \longrightarrow \mathrm{End}_k(M).$$

Then M is called an $A - G$-module if for every $g \in G$, $\nabla_g(ma) = \nabla_g(m)\nabla_g(a)$.

Definition 6. *An additive mapping $\phi : M \longrightarrow N$ where M and N are $A-G$-modules is called g-linear, $g \in G$, if $\phi(ma) = \phi(m)\nabla_g(A)$.*

Lemma 8. *A g-linear morphism $\phi : M \longrightarrow N$ is determined by its values on a set of generators. Moreover, given the composition $M \xrightarrow{\phi} N \xrightarrow{\psi} P$ where M, N, P are (right) A-modules. If one of ϕ, ψ are g-linear, the other A-linear, then the composition $\phi \circ \psi$ is g-linear.*

Proof. $\phi(\sum_{i\in I} m_i a_i) = \sum_{i\in I} \phi(m_i a_i) = \sum_{i\in I} \phi(m_i)\nabla_g(a_i)$ so that indeed the morphism is determined by $\phi(m_i)$, $m_i \in I$. Given the composition. If ϕ is g-linear, ψ is A-linear, then $\psi(\phi(m \cdot a)) = \psi(\phi(m) \cdot \nabla_g(a)) = \phi \circ \psi(m) \cdot \nabla_g(a)$.

If ψ is g-linear, ϕ is A-linear, then $\psi(\phi(ma)) = \psi(\phi(m) \cdot a) = \psi(\phi(m)) \cdot \nabla_g(a) = \phi \circ \psi(m) \cdot \nabla_g(a)$. □

Definition 7. *An A-linear morphism $\phi : M \longrightarrow N$ between two $A-G$-modules is called a morphism (of $A-G$-modules) if $\phi(\nabla_g(m)) = \nabla_g(\phi(m))$ for all $g \in G$, $m \in M$.*

Lemma 9. *Given a morphism of $A-G$-modules $\phi : M \longrightarrow N$. Then* $\ker\phi$, $\operatorname{Im}\phi$, $\operatorname{Coker}\phi$ *are all $A-G$-modules.*

Proof. Because the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ {\scriptstyle\nabla_g}\downarrow & & \downarrow{\scriptstyle\nabla_g} \\ M & \xrightarrow{\phi} & N \end{array}$$

commutes, it follows that $\nabla_g|_{\ker\phi} : \ker\phi \longrightarrow \ker\phi$ and that $\ker\phi$ inherits a structure of $A-G$-module. Accordingly, $\operatorname{Im}\phi$ is an $A-G$-submodule. The quotient $N/\operatorname{Im}\phi$ is an $A-G$-module in the obvious way. □

Lemma 10. *Let M, N be two $A-G$ modules. Then $\operatorname{Hom}_A(M,N)$ is an $A-G$ module by $\nabla_g(\phi) = \nabla_{g^{-1}} \circ \phi \circ \nabla_g$. Furthermore,*

$$\operatorname{Hom}_{A-G}(M,N) = \operatorname{Hom}_A(M,N)^G.$$

Proof.

$$\begin{aligned} \nabla_g(\phi a)(m) &= \nabla_{g^{-1}} \circ \phi a \circ \nabla_g(m) = \nabla_g(\phi a(\nabla_{g^{-1}}(m))) \\ &= \nabla_g(\phi(\nabla_{g^{-1}}(m))a) = \nabla_g(\phi(\nabla_{g^{-1}}(m)))\nabla_g(a) \\ &= \nabla_g(\phi)(m)\nabla_g(a) = \nabla_g(\phi)\nabla_g(a)(m). \end{aligned}$$

Thus $\nabla_g(\phi a) = \nabla_g(\phi)\nabla_g(a)$. Also, $\phi \in \operatorname{Hom}_{A-G}(M,N) \Rightarrow \phi \in \operatorname{Hom}_A(M,N)$ and $\nabla_g \circ \phi = \phi \circ \nabla_g \Leftrightarrow \phi = \nabla_{g^{-1}} \circ \phi \circ \nabla_g$. □

We will use the following definition of reductive:

Definition 8. *The unipotent radical $R_u(G)$ of G is the maximal closed, connected, unipotent, normal subgroup of G. G is reductive if $R_u(G)$ is trivial.*

Lemma 11. *If G is reductive, then the category of $A-G$-modules has enough projectives.*

Proof. Consider the $A-G$ module M. As A-module M has a free A-module mapping onto it. For each $g \in G$ we can lift ∇_g^M to ∇_g as shown in the diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & M & \xleftarrow{\ \varepsilon\ } & A^n \\ & & \Big\downarrow{\scriptstyle \nabla_g^M} & & \Big\downarrow{\scriptstyle \nabla_g} \\ 0 & \longleftarrow & M & \xleftarrow[\ \varepsilon\ ]{} & A^n \end{array}$$

If G is reductive, then $\nabla_{g_1} \circ \nabla_{g_2} = \nabla_{g_1 g_2}$ Thus A^n is an $A-G$ module under this action. To prove that A^n is projective with respect to this $A-G$ action we notice that because A^n is A-projective, there exists an A-linear γ as shown in the diagram below:

$$\begin{array}{ccccc} & & A^n & \xleftarrow{\ \nabla_g\ } & A^n \\ & \swarrow{\scriptstyle \gamma} & \Big\downarrow{\scriptstyle \psi} & & \\ N & \xrightarrow[\ \phi\ ]{} & Q & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \nabla_g} & & \Big\downarrow{\scriptstyle \nabla_g} & & \\ N & \xrightarrow[\ \phi\ ]{} & Q & \longrightarrow & 0 \end{array}$$

If G is reductive, this γ can be chosen so that it commutes with ∇_g. We prove this for $G = G_m = k^*$. Then because ϕ is an $A-G$ module homomorphism, $\ker\phi$ is an $A-G$ module and so for $n \in \ker\phi$, $\nabla_g(n) = \alpha n$ for some $n \in k^*$. Thus $\gamma(\nabla_g(e_i)) = \gamma(\alpha e_i) = \alpha\gamma(e_i)$, $\nabla_g(\gamma(e_i)) = \beta\gamma(e_i)$ for some β. Finally, because $0 = \phi(\gamma(\nabla_g(e_i)) - \nabla_g(\gamma(e_i)) = (\beta - \alpha)\psi(e_i)$, $\alpha = \beta$ whenever necessary ☐

Proposition 2. *Let M, N be two $A-G$ modules where G is reductive. Then*

$$\mathrm{Ext}^i_{A-G}(M, N) \cong \mathrm{Ext}^i_A(M, N)^G.$$

Proof. Choose locally free resolutions and lift the g-action for each $g \in G$:

$$\begin{array}{ccccccccccccc} 0 & \longleftarrow & M & \longleftarrow & A^{n_0} & \longleftarrow & A^{n_1} & \longleftarrow & \cdots & \longleftarrow & A^{n_i} & \longleftarrow & \cdots \\ & & \Big\downarrow{\scriptstyle \nabla_g} & & \Big\downarrow{\scriptstyle \nabla_{g,0}} & & \Big\downarrow{\scriptstyle \nabla_{g,1}} & & & & \Big\downarrow{\scriptstyle \nabla_{g,i}} & & \\ 0 & \longleftarrow & M & \longleftarrow & A^{n_0} & \longleftarrow & A^{n_1} & \longleftarrow & \cdots & \longleftarrow & A^{n_i} & \longleftarrow & \cdots \end{array}$$

Then

$$\mathrm{Ext}^i_{A-G}(M, N) \cong h^i(\mathrm{Hom}_{A-G}(M, \cdot)) \cong \mathrm{Ext}^i_A(M, N)^G,$$

where $\mathrm{Ext}^i_A(M, N)$ is an $A-G$ module by the previous lemmas. Also notice that the action on the Yoneda representation follows. ☐

Now, the definition of $\mathcal{O}_X - G$ modules on a scheme X/k is clear, and that the closure of any G-orbit is an $\mathcal{O}_X - G$ module. Thus classifying closure of G-orbits is equivalent to classifying $A - G$ modules. This will be clear and exemplified in the following.

4.3 Invariant Theory of Bundles

Before we can make any computations at all, we must find a category of $A-G$-modules for a ring A and a reductive group G such that the quotient exists and corresponds to, or at least contains $\mathcal{M}^{ss}_{X,\mathcal{L}}(r; c_1, \ldots, c_r)$ as a subscheme. References for this can be found in Seshadri[11], Gieseker[3],Maruyama[6],[7]. More references to the applications of bundles can be found in GIT[2]. The two main methods for studying bundles on projective spaces by geometric invariant theory, are the following:

A. Choose a large number of points $\{P_1, \ldots, P_N\} \subseteq X$ and associate to each rank r vector bundle $\mathcal{E}$ on X the family of linear maps

$$\gamma_i : \Gamma(X, \mathcal{E}(n)) \longrightarrow \mathcal{E}(n)(P_i) \longrightarrow 0, \ (n >> 0),$$

$\gamma_i(s) = s(P_i)$. Equivalently this gives N subspaces of codimension r

$$W_i = \{s \in \Gamma(X, \mathcal{E}(n)) | s(P_i) = 0\}.$$

Let $G = \mathrm{Sl}(\Gamma(X, \mathcal{E}(n)))$. If $\mathcal{E}$ is stable and n and P_i are sufficiently chosen, then $\mathbb{P}(W_i) \subseteq \mathbb{P}(\Gamma(X, \mathcal{E}(n)))$ is G-stable.

B. Suppose a line bundle $\mathcal{L}$ on X is given. To each pair $(\mathcal{E}, \phi)$ consisting of a rank r bundle $\mathcal{E}$ and $\phi : \wedge^r \mathcal{E} \xrightarrow{\cong} \mathcal{L}$ we associate the canonical morphism

$$\wedge^r \Gamma(X, \mathcal{E}(n)) \longrightarrow \Gamma(X, \mathcal{L}(nr)).$$

Choosing a basis this gives $\Gamma(X, \mathcal{L}(nr)) \cong k^M$ and thus $\Gamma(X, \mathcal{L}(nr))^\vee \longrightarrow (\wedge^r \Gamma(X, \mathcal{E}(n)))^\vee$ gives M elements $\omega_1, \ldots, \omega_M \in (\wedge^r \Gamma(X, \mathcal{E}(n)))^\vee$. If $\mathcal{E}$ is stable, $(\omega_1, \ldots, \omega_M)$ is stable with respect to $G = \mathrm{Sl}(\Gamma(X, \mathcal{E}(n)))$.

In case A we can study moduli of sequences of linear subspaces of $\mathbb{P}(V)$ under the action of G. Here we get

$$\mathcal{M}^s(r; c_1, \ldots, c_r) \hookrightarrow (\mathrm{Grass})^N / G.$$

In case B we can study the moduli of representations associated to sequences of linear morphisms. Then we have

$$\mathcal{M}^s(r; c_1, \ldots, c_r) \hookrightarrow \underline{\mathrm{repr}} / G.$$

In the following, these moduli spaces will be studied.

5 The Grassmanian Scheme

We know that the Grassmanian functor is representable, and we know that the Grassmanian scheme can be embedded in $\mathbb{P}^n$ for some n. The action of $\mathrm{Gl}(n)$ on $\mathbb{P}^n$ needs a linearization of an invertible sheaf, and then we can study the action of $\mathrm{Gl}(n)$ on Grass by the corresponding action on the plücker coordinates. In this chapter we will explain how the theory of formal moduli of $A-G$-modules can be used to simplify this setup.

5.1 Projective $(n-1)$-space as formal moduli

As a set, we have that $\mathbb{P}^{n-1}$ corresponds bijectively to $k^n - \{0\}/k^*$. However, this geometric quotient does not exist as an algebraic scheme because the points are not stable under the action of the reductive group k^*. The problem of non stability is solved in the non commutative case by adding more generic points. Here the only extra generic point is at infinity, and we want to exclude it from our computation. Thus we chose a worst point, we compute the local formal moduli and hope that an algebraization of the local formal moduli and its versal family will give us some open neighborhood in the moduli, and that we will be able to glue an open covering to a complete moduli.

We consider the affine n-space under the action of the reductive group $G = k^*$. Consider the point $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in k^n$, that is a 1 at the i'th. place and 0 elsewhere. The orbit of this point is given by the $A-G$ module

$$V_i = k[\underline{x}]/(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Then this computation is justified by lemma 4 because as the modules in question has support strictly inside an open affine.

Put $\mathfrak{a}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $A = k[\underline{x}]$. We have that $\mathrm{Ext}^1_{A-G}(V_i, V_i) \cong \mathrm{Ext}^1_A(V_i, V_i)^G \cong \mathrm{Hom}_k(\mathfrak{a}_i, A/\mathfrak{a}_i)^G$. Thus a $\phi \in \mathrm{Hom}_k(\mathfrak{a}_i, A/\mathfrak{a}_i)$ is in $\mathrm{Ext}^1_{A-G}(V_i, V_i)$ if and only if it is invariant for all $g \in G$:

$$\mathfrak{a}_i \xrightarrow{\nabla_g} \mathfrak{a} \xrightarrow{\phi} A/\mathfrak{a} \xrightarrow{\nabla_{g^{-1}}} A/\mathfrak{a}.$$

As ϕ is determined by its action on the generators, and as there are no relations on the generators in $\mathfrak{a}$, we get a basis for $\mathrm{Ext}^1_{A-G}(V_i, V_i)$ consisting of the morphisms ϕ_j,

$$\phi_j(x_p) = \begin{cases} x_i, & \text{p=j} \\ 0, & p \neq j \end{cases}$$

Remark 2. *Notice that when G is reductive,* $\mathrm{Ext}^1_{A-G}(V_i, V_i) \cong \mathrm{Ext}^1(V_i, V_i)^G$. *This can be seen for example by the fact that* $\mathrm{Spec}(A)/G = \mathrm{Spec}(A^G)$ *when A is a finite type k-algebra, k algebraically closed and G reductive, [2].*

As the projective n-dimensional space is smooth, we know that the local formal moduli will be formally smooth. We are looking for an alternative way of computing with $\mathbb{P}^n$, and thus we do the computation even if we do know the answer.

Consider the Koszul-complex of V_i. As A is regular, this complex is a free resolution. We compute the cup and Massey products. We do the computation for $V_1 = V$. The case with V_i, $1 \leq i \leq n$ follows by symmetry.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & V & \longleftarrow & A & \xleftarrow{(x_2,\ldots,x_n)} & A^{n-1} & \xleftarrow{d_2} & A^r & \longleftarrow & , \\ & & & & & \swarrow & & \swarrow & & & \\ 0 & \longleftarrow & V & \longleftarrow & A & \xleftarrow{(x_2,\ldots,x_n)} & A^{n-1} & \longleftarrow & A^r & \longleftarrow & \end{array}$$

where d_2 is given by the following matrix

$$d_2 = \begin{pmatrix} x_3 & x_4 & \cdots & x_n & \cdots & 0 \\ -x_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & -x_2 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & x_n \\ 0 & 0 & \cdots & -x_2 & \cdots & -x_{n-1} \end{pmatrix}$$

Then a translation of the basis for $\operatorname{Ext}^1_{A-G}$ to the Yoneda complex is given by $\{(\alpha_{e_i,1}, \alpha_{e_i,2})\}_{i=1}^{i=n-1}$ where $\alpha_{e_i,1} = x_1 e_i$ and where $\alpha_{e_1,2}$ is given by the equation

$$x_1 e_i d_2 = -\sum_{j=2}^{n} x_j e_{j-1} \alpha_{e_i,2}.$$

It is straight forward to compute that the cup products are all identically zero, and so an obvious algebraization of $\hat{H}_V$ is

$$H_V = k[t_1, ..., t_{n-1}].$$

The versal family is then given by

$$\tilde{M}(t_1, ..., t_{n-1}) = (x_2 \otimes_k 1 + x_1 \otimes_k t_1, x_3 \otimes_k 1 + x_1 \otimes_k t_2, \ldots, x_n \otimes 1 + x_1 \otimes_k t_{n-1}).$$

This parameterizes all zero sets of $(x_2 + t_1 x_1, x_3 + t_2 x_1, \ldots, x_n + t_{n-1} x_1)$, i.e. all the lines

$$(x_1, a_1 x_1, a_2 x_1, \ldots, a_{n-1} x_1) = x_1(1, a_1, a_2, \ldots, a_{n-1}).$$

This is obviously the open subset D_{x_1} of projective $n-1$-space, and gluing the the local moduli and versal family in their intersections for $V_1, ..., V_n$, we get exactly $\mathbb{P}^{n-1}$.

5.2 Projective $(n-1)$-space as noncommutative scheme

we start by recalling the definition of Jacobson Topology:

Definition 9. *Let $\underline{c}$ be a diagram of A-modules. We define the* **Jacobson topology** on $\underline{c}$ as the topology generated by

$$D(a) = \{c \in \mathrm{ob}(\underline{c}) | \rho(a) \in \mathrm{End}(c) \text{ is injective } \}$$

when $a \in A$ and $\rho : A \longrightarrow \mathrm{End}(c)$ is the A-module structure.

$\rho(a) : c \longrightarrow c$ is injective is the same as $a \notin \mathrm{Ann}(c)$, and for simple modules this is equivalent with $\rho(a)$ being an isomorphism. This is in line with [9].

The Jacobson topology on a diagram of A-modules is in fact defined by $\rho(a)$ an isomorphism. Our objects here are the orbits, and the orbits in $D(a)$ is exactly those where $G \times \{x\} \cong G$. that is $\mathrm{Spec}(k[x_i, x_i^{-1}]) \times \mathrm{Spec}\, k \cong \mathrm{Spec}(k[x_i, x_i^{-1}])$. This correspond to our $D(x_i)$ defined above. The reason for the different definition is that in our case we study the closure of the orbits instead of the actual orbits.

Let $\underline{c}$ be the set of lines through 0 in $\mathbb{A}^n$. That is all the orbits of $\mathbb{A}^n - \{0\}$ under the action of $G = k^*$.

Let $A = k[x_1, \ldots, x_n]$. Then if L is a line through $(a_1, \ldots, a_n)$ with $a_1 \neq 0$, the orbit (or line) is given by the quotient

$$A/(x_2 - \frac{a_2}{a_1}x_1, x_3 - \frac{a_3}{a_1}x_1, \ldots, x_n - \frac{a_n}{a_1}x_1).$$

It is then easy to see that all the modules on the form

$$A/(x_1 - a_1x_i, x_2 - a_2x_i, \cdots, \hat{x}_i, \ldots, x_n - a_{n-1}x_i)$$

are in $D(x_i)$. The computation in the previous sections then proves that $\mathcal{O}(D(x_i)) \cong k[t_1, \ldots, t_{n-1}]$, because the versal family covers all of $D(x_i)$ exactly once. This proves that $\mathbb{P}^{n-1}$ is a scheme for the lines through 0 on $\mathbb{A}^n$, and it is not affine. Also, the limit $\mathcal{O}(\underline{c}) = \varinjlim_i \mathcal{O}(D(x_i)) \cong k$ because $A_{x_1x_2x_3\cdots x_n} \cong k$, k algebraically closed.

5.3 Global computation of $\mathbb{P}^n$

We have computed $\mathbb{P}^n$ as an affine quotient, i.e. we have identified $\mathbb{P}^n$ with $(\mathbb{A}^{n+1}/k^*) - \{0\}$. Now $\mathbb{P}^n$ is in fact not an affine quotient, $\mathbb{P}^n \cong (\mathbb{A}^{n+1} - \{0\})/k^*$, and it is not trivial that these two are the same. From our computation however, it follows that the affine computation is enough, but here we will use the global computation to understand why.

Let $A = k[x_0, \ldots, x_n]$,

$$X = \mathrm{Spec}(A) - \{0\} = \bigcup_{i=0}^{n} D(x_i) = \bigcup_{U \in \mathcal{U}} U$$

. We put $M_i = A/(x_0, \ldots, \hat{x}_i, \ldots, x_n)$, $\mathcal{F}_i = \tilde{M}_i|_X, i = 1, \ldots, n$, and in this particular computation we may put $M = M_0$ for simplicity. Then we find that the exact sequence

$$0 \longleftarrow M \longleftarrow \tilde{A} \xleftarrow{(x_1,\ldots,x_n)} \tilde{A}^{n-1} \xleftarrow{d_2} \tilde{A}^r \longleftarrow \cdots,$$

with

$$d_2 = \begin{pmatrix} x_2 & x_3 & \cdots & x_n & \cdots & 0 \\ -x_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & -x_1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & x_n \\ 0 & 0 & \cdots & -x_1 & \cdots & -x_{n-1} \end{pmatrix}$$

restricts to the exact sequence on X:

$$0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{O}_X \longleftarrow \mathcal{O}_X^{n-1} \longleftarrow \mathcal{O}_X^r \longleftarrow \cdots.$$

As usual, we put $\check{C}^p(\mathcal{U}\,\mathcal{H}\mathrm{om}(\mathcal{L}_q, \mathcal{F})) = \check{C}^{p,q}$, and we denote by C^n the total complex associated to $\check{C}^{p,q}$. Then we have that

$$\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{F}) = H^i(C^n).$$

We find

$$\begin{aligned} \check{C}^{0,0} &= \oplus_{i=0}^n \mathcal{H}\mathrm{om}(\mathcal{L}_0, \mathcal{F})(D(x_i)) = \oplus_{i=0}^n \mathrm{Hom}(A_{x_i}, (A/(x_1, \ldots, x_n))_{x_i} \\ &= \oplus_{i=0}^n (A/(x_1, \ldots, x_n)_{x_i} = A_{x_0}/(x_1, \ldots, x_n). \end{aligned}$$

$$\begin{aligned} \check{C}^{1,0} &= \oplus_{i_0<i_1} \mathcal{H}om(\mathcal{L}_0, \mathcal{F})(D(x_{i_0}x_{i_1})) = \oplus_{i_0<i_1} \mathrm{Hom}(A_{x_{i_0}x_{i_1}}, (A/(x_1, \ldots, x_n))_{x_{i_0}x_{i_1}}) \\ &= \oplus_{i_0<i_1} (A/(x_1, \ldots, x_n))_{x_{i_0}x_{i_1}} = 0. \end{aligned}$$

$$\begin{aligned} \check{C}^{0,1} &= \oplus_{i=0}^n \mathcal{H}\mathrm{om}(\mathcal{L}_1, \mathcal{F})(D(x_i)) = \oplus_{i=0}^n \mathrm{Hom}(A_{x_i}^{n-1}, (A/(x_1, \ldots, x_n))^{n-1} \\ &= \oplus_{i=0}^n (A/(x_1, \ldots, x_n)_{x_i})^{n-1} = (A_{x_{i_0}}/(x_1, \ldots, x_n))^{n-1}. \end{aligned}$$

For exactly the same reasons, $\check{C}^{2,0} = \check{C}^{1,1}$ and

$$\check{C}^{0,2} = (A_{x_0}/(x_1, \ldots, x_n))^r.$$

Then the total complex is

$$A_{x_0}/(x_1, \ldots, x_n) \overset{(x_1,\ldots,x_n)^T}{\longrightarrow} (A_{x_0}/(x_1, \ldots, x_n))^{n-1} \xrightarrow{s_2^T} (A_{x_0}/(x_1, \ldots, x_n))^r,$$

and $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F}) \cong A_{x_0}/(x_1, \ldots, x_n)$. Taking the G-action into account, this is exactly as in the affine case. Thus we are through. Notice that globally or locally we classify the orbits. Thus the two moduli spaces of orbits must be the same.

5.4 The Grassmanian Scheme $\mathrm{Grass}(2,3)$

We start by computing this special case. Then we will compute $\mathrm{Grass}(r,n)$ in general later.

A 2-dimensional subspace of k^3 is given by two linearly independent vectors. That is a maximal rank 3×2-matrix A. A subspace is invariant under the usual column operations, that is under the action of $\mathrm{Gl}(2)$. Thus we are considering the quotient $k[x_{ij}]_{\substack{1\leq i\leq 3\\ 1\leq j\leq 2}}/\,\mathrm{Gl}(2)$.

Letting

$$M_1=\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix},\ M_2=\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix},\ M_3=\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix},$$

we can describe the closure of the orbits of these subspaces as

$$V_1=o^-(M_1)=A/(x_{31},x_{32}),\ V_2=o^-(M_2)=A/(x_{21},x_{22}),\ V_3=o^-(M_3)=A/(x_{11},x_{12}),$$

where we have put $A=k[x_{ij}]_{\substack{1\leq i\leq 3\\ 1\leq j\leq 2}}$. By symmetry, it is sufficient to compute the formal moduli with its versal family for one of the orbits. Thus we consider $V=V_1$ with Koszul resolution

$$\begin{array}{ccccccccc}
0 & \longleftarrow & V & \xleftarrow{(x_{31},x_{32})} & A & \xleftarrow{\begin{pmatrix}-x_{32}\\x_{31}\end{pmatrix}} & A^2 & \longleftarrow & A \longleftarrow 0\\
 & & & & & \swarrow & & \swarrow & \\
0 & \longleftarrow & V & \xleftarrow[(x_{31},x_{32})]{} & A & \xleftarrow[\begin{pmatrix}-x_{32}\\x_{31}\end{pmatrix}]{} & A^2 & \longleftarrow & A \longleftarrow 0
\end{array}$$

Now we consider the action of $G=\mathrm{Gl}(2)$ given by

$$(x_{31},x_{32})\xrightarrow{\nabla_g}(x_{31},x_{32})\xrightarrow{\phi}A/(x_{31},x_{32})\xrightarrow{\nabla_{g^{-1}}}A/(x_{31},x_{32})\ .$$

We find a basis for $\mathrm{Ext}^1_{A-G}(V,V)\cong\mathrm{Ext}^1(V,V)^G$ the following way:
First we notice that the action of G is given by

$$\begin{pmatrix}x_{11}&x_{21}&x_{31}\\x_{12}&x_{22}&x_{32}\end{pmatrix}\mapsto g\cdot\begin{pmatrix}x_{11}&x_{21}&x_{31}\\x_{12}&x_{22}&x_{32}\end{pmatrix}$$

where $g\in G=\mathrm{Gl}(2)$, and where we have transposed the matrix of the Grassmanian to be able to multiply from the left.

Put $\phi = (f_{31}, f_{32})$. We consider the the conditions on ϕ under the action of the generators of G. Firstly, ϕ must be invariant under $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. Thus f_{31} and f_{32} must be homogeneous of degree 1. That is

$$\begin{aligned} f_{31} &= \alpha_{11}x_{11} + \alpha_{21}x_{21} + \alpha_{12}x_{12} + \alpha_{22}x_{22}, \\ f_{32} &= \beta_{11}x_{11} + \beta_{21}x_{21} + \beta_{12}x_{12} + \beta_{22}x_{22}. \end{aligned}$$

Secondly, ϕ must be invariant under $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. this gives

$$f_{31} = \nabla_{g^{-1}}(f_{32}) = \beta_{11}x_{12} + \beta_{21}x_{22} + \beta_{12}x_{11} + \beta_{22}x_{21}$$

and similarly for f_{32} Then

$$\begin{aligned} f_{31} &= \alpha_{11}x_{11} + \alpha_{21}x_{21} + \alpha_{12}x_{12} + \alpha_{22}x_{22} \\ f_{32} &= \alpha_{12}x_{11} + \alpha_{22}x_{21} + \alpha_{11}x_{12} + \alpha_{21}x_{22}. \end{aligned}$$

The last generators are $g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $c \neq 0$. We compute the action for the first of these, the second is similar.

ϕ maps to

$$(x_{31}, x_{32}) \mapsto (x_{31}, x_{32}+cx_{31}) \mapsto (f_{31}, f_{32}+cf_{31}) \mapsto (\nabla_{g^{-1}}(f_{31}), \nabla_{g^{-1}}(f_{32}+cf_{31})).$$

The first condition gives

$$\begin{aligned} \alpha_{11}x_{11} + \alpha_{21}x_{21} + \alpha_{12}x_{12} + \alpha_{22}x_{22} &= \\ (\alpha_{11} - \alpha_{12}c)x_{11} + (\alpha_{21} - \alpha_{22}c)x_{21} + \alpha_{12}x_{12} + \alpha_{22}x_{22} &\Rightarrow \\ \alpha_{12} = \alpha_{22} &= 0. \end{aligned}$$

Thus

$$\begin{aligned} f_{31} &= \alpha_{11}x_{11} + \alpha_{21}x_{21} \\ f_{32} &= \alpha_{11}x_{12} + \alpha_{21}x_{22}. \end{aligned}$$

Then the second condition gives

$$\begin{aligned} \alpha_{11}x_{12} + \alpha_{21}x_{22} &= \nabla_{g^{-1}}(\alpha_{11}x_{12} + \alpha_{21}x_{22} + \alpha_{11}cx_{11} + \alpha_{11}cx_{21} \\ &= \alpha_{11}(x_{12} - cx_{11}) + \alpha_{21}(x_{22} - cx_{21}) + \alpha_{11}cx_{11} + \alpha_{21}cx_{21} \\ &= \alpha_{11}x_{12} + \alpha_{21}x_{22}, \end{aligned}$$

which is already fulfilled.

All in all we might write

$$\phi = (f_{31}, f_{32}) = (\alpha_{11}x_{11}+\alpha_{21}x_{21}, \alpha_{11}x_{12}+\alpha_{21}x_{22}) = \alpha_{11}(x_{11}, x_{12})+\alpha_{21}(x_{21}, x_{22}).$$

We have computed $\mathrm{ext}^1_{A-G}(V,V)=2$, and a basis is given by

$$\alpha_{e_1,1}=(x_{11},x_{12}),\ \alpha_{e_1,2}=\begin{pmatrix}-x_{12}\\ x_{11}\end{pmatrix}$$
$$\alpha_{e_2,1}=(x_{21},x_{22}),\ \alpha_{e_2,2}=\begin{pmatrix}-x_{22}\\ x_{21}\end{pmatrix}.$$

The cup-products are all identically zero, and so the (algebraic) local moduli is $k[t_1,t_2]$. The local versal family is given by

$$d(t_1,t_2)=(x_{31}\otimes 1+x_{11}\otimes t_1+x_{21}\otimes t_2, x_{32}\otimes 1+x_{12}\otimes t_1+x_2\otimes t_2),$$

with geometric points:

$$\begin{matrix}x_{31}+a_1x_{11}+a_2x_{21}=0\\ x_{32}+a_1x_{12}+a_2x_{22}=0\end{matrix}\Rightarrow\begin{matrix}x_{31}=-a_1x_{11}-a_2x_{21}\\ x_{32}=-a_1x_{12}-a_2x_{22}\end{matrix}\Rightarrow$$

$$\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\end{pmatrix}=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\\ -a_1x_{11}-a_2x_{21} & -a_1x_{12}-a_2x_{22}\end{pmatrix}=$$

$$x_{11}\begin{pmatrix}1 & 0\\ 0 & 0\\ -a_1 & 0\end{pmatrix}+x_{12}\begin{pmatrix}0 & 1\\ 0 & 0\\ 0 & -a_1\end{pmatrix}+x_{21}\begin{pmatrix}0 & 0\\ 1 & 0\\ -a_2 & 0\end{pmatrix}+x_{22}\begin{pmatrix}0 & 0\\ 0 & 1\\ 0 & -a_2\end{pmatrix},$$

which gives the two dimensional subspace corresponding to that point. This is not the form we would like. This is solved in the next subsection, where we compute the general Grassmanian.

Notice that this and the next computation is justified by Lemma 4 because all the modules in question has support strictly inside an open affine subset.

5.5 The General Grassmanian scheme $\mathrm{Grass}(r,n)$

The method illustrated in the previous section, illustrates the general case. However, there is a more straight forward way to find the scheme $\mathrm{Grass}(r,n)$, which will be developed here.

Put

$$A=k[x_1,...,x_n],\ V_{i_1,\dots,i_r}=A/\mathfrak{a}_{i_1,\dots,i_r}$$

where $1\leq i_1<...<i_r\leq n$ and $\mathfrak{a}=(x_1,...,\hat{x}_{i_1},...,\hat{x}_{i_r},...,x_n)$, i.e. the ideal generated by all $x_1,...,x_n$ except $x_{i_1},..,x_{i_r}$. By symmetry it is enough consider $V=V_{1,2,\dots,r}$. Exactly as in the case with $\mathbb{P}^{n-1}$, $k^*=G$ acts on A, and V is an $A-G$-module. Thus we can compute $\hat{H}(V)$ and its algebraization $H(V)$.

To find a basis for $\mathrm{Ext}^1_{A-G}(V,V)$, we consider

$$(x_{r+1},...,x_n)\xrightarrow{\nabla_g}(x_{r+1},...,x_n)\xrightarrow{\phi}A/(x_{r+1},...,x_n)\xrightarrow{\nabla_{g^{-1}}}A/(x_{r+1},...,x_n).$$

It then follows that ϕ is invariant under k^* if and only if

$$\phi = (f_{r+1}, \ldots, f_n)$$

where $f_j = \sum_{i=1}^{r} \alpha_i x_i$, i.e. homogeneous of degree 1. Thus $\dim_k \operatorname{Ext}^1_{A-G}(V,V) = r \cdot (n-r)$. The Koszul complex is

$$\begin{array}{ccccccccc}
0 & \longleftarrow & V & \longleftarrow & A & \xleftarrow{(x_{r+1},\ldots,x_n)} & A^{n-r} & \xleftarrow{d_2} & A^s & \longleftarrow & \cdots \\
 & & & & & \swarrow & & \swarrow & & & \\
0 & \longleftarrow & V & \longleftarrow & A & \xleftarrow{(x_{r+1},\ldots,x_n)} & A^{n-r} & \xleftarrow{d_2} & A^s & \longleftarrow & \cdots
\end{array}$$

where $s = \binom{n-r}{2}$ and

$$d_2 = \begin{pmatrix}
x_{r+1} & x_{r+2} & \cdots & x_n & 0 & 0 & \cdots & 0 \\
-x_r & 0 & \cdots & 0 & x_{r+2} & x_{r+3} & \cdots & \vdots \\
0 & -x_r & \cdots & 0 & -x_{r+1} & 0 & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & -x_{r+1} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & x_n \\
0 & 0 & \cdots & -x_r & 0 & 0 & \cdots & -x_{n-1}
\end{pmatrix}.$$

We find that a basis for $\operatorname{Ext}^1_{A-G}(V,V)$ in the Yoneda complex is given by

$$\alpha_{ij} = \begin{cases} \alpha_{ij,1} \\ \alpha_{ij,2} \end{cases}$$

where $\alpha_{ij,1} = x_i e_j$, $1 \leq i \leq r$, $1 \leq j \leq n-r$ and $\alpha_{ij,2}$ is given by the condition

$$\alpha_{ij,1} \cdot d_2 = (x_{r+1}, \ldots, x_n) \cdot \alpha_{ij,2}.$$

We compute and find that all the cup products are identically zero, and so

$$H(V) \cong k[t_{ij}]_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n-r}},$$

and the versal family is given by $\tilde{M}/(t_{ij}) =$

$(x_{r+1} \otimes 1 + x_1 \otimes t_{11} + \cdots + x_r \otimes t_{r1}, \ldots, x_n \otimes 1 + x_1 \otimes t_{1,n-r} + x_2 \otimes t_{2,n-r} + \cdots + x_r \otimes t_{r,n-r})$.

The geometric points of this scheme are the zero sets

$$\begin{aligned}
x_{r+1} &= -t_{11}x_1 - \cdots - t_{r,1}x_r \\
&\vdots \\
x_n &= -t_{1,n-r}x_1 - \cdots - t_{r,n-r}x_r.
\end{aligned}$$

which gives the points

$$x_1(1,\dots,-t_{11},-t_{12},\dots,-t_{1,n-r})+x_2(0,1,\dots,-t_{21},-t_{22},\dots,-t_{2,n-r})$$
$$+\cdots+x_r(0,\dots,1,-t_{r,1},-t_{r,2},\dots,-t_{r,n-r}),$$

i.e. the subspace spanned by these vectors. This proves that we can glue the $\binom{n}{r}$ affine schemes $\mathrm{Spec}(H(V_{i_1,\dots,i_r}))$ to get $\mathrm{Grass}(r,n)$.

Notice that the sets $H(V_{i_1,\dots,i_r})$ corresponds to $D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_r})$, the open sets in the Jacobson topology, proving that the prescheme $H(V_{i_1,\dots,i_r})$ is a scheme for the Grassmanian. This is as it must be from general results.

Lemma 12.
$$\mathrm{Grass}(r,n)=\bigsqcup \mathrm{Spec}(H(V_{i_1,\dots,i_r})).$$

□

Because the Grassmanian is used to prove the projectivity of various moduli spaces, the embedding of $\mathrm{Grass}(r,n)$ in $\mathbb{P}^{s-1}$ for $s=\binom{n}{r}$ is of importance. Let

$$v_1=(1,\dots,-t_{11},\dots,t_{1,n-r}),\dots,$$
$$v_r=(0,\dots,1,-t_{r,1},\dots,-t_{r,n-r}).$$

Then

$$v_1\wedge v_2\wedge\cdots\wedge v_r=e_1\wedge\cdots\wedge e_r+\sum_{i=2}^{\binom{n}{r}}u_i(t_{ij})p_j$$

where the p_i's are the plücker coordinates, and thus polynomials in $\underline{t}$.

Sending p_i to u_i gives a surjection $k[p_1,\dots,p_s]\twoheadrightarrow k[t_{ij}]$ that respects the versal families. This gives the plücker embedding

$$\mathrm{Grass}(r,n)\hookrightarrow\mathbb{P}^{s-1}.$$

We hope that this discussion will give a way to study the N-subspace problem intrinsic (without the projective embedding), i.e. without choosing a Sl(3)-linearization of an invertible sheaf.

5.6 The action of $G=\mathrm{Gl}(n)$ on $\mathrm{Grass}(r,n)$

We are going to use the Grassmanian in the following way: Let X/k be an algebraic scheme, $\mathcal{M}(x;r)=\{\text{Rank } r\text{-bundles}\}/\cong$.

Choose $n>>0$ and $P_1,\dots,P_N\in X$. For $\mathcal{E}\in\mathcal{M}(r)$ we have

$$0\longrightarrow W_i\longrightarrow\Gamma(X,\mathcal{E}(n))\longrightarrow\mathcal{E}(n)(P_i)\longrightarrow 0$$

which gives N subspaces of codimension r where $n=\dim_k\Gamma(X,\mathcal{E}(n))$. We get

$$\phi:\mathcal{M}(x;r)\longrightarrow\mathrm{Grass}(n-r,n)^n.$$

For $\mathcal{E} \cong \mathcal{F}$ we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_i^{\mathcal{E}} & \longrightarrow & \Gamma(X, \mathcal{E}(n)) & \longrightarrow & \mathcal{E}(n) & \longrightarrow & 0 \\
 & & \iota \Big\uparrow \cong & & \gamma \Big\uparrow \cong & & \Big\uparrow \cong & & \\
0 & \longrightarrow & W_i^{\mathcal{F}} & \longrightarrow & \Gamma(X, \mathcal{F}(n)) & \longrightarrow & \mathcal{F}(n) & \longrightarrow & 0
\end{array}$$

where ι is induced by γ, i.e. the diagram commutes. Thus the isomorphism classes in $\mathcal{M}(X; r)$ corresponds to the orbits in $\mathrm{Grass}(r, n)^n$ under the $\mathrm{Gl}(n)$-action induced by the action on $\mathrm{Grass}(r, n)$ given by $\phi \mapsto g \cdot \phi$. It is so because given

$$k^r \xrightarrow{\phi} k^n.$$

This is sent to $g\phi$ given in the diagram

$$\begin{array}{ccc}
k^r & \xrightarrow{\phi} & k^n \\
 & \searrow^{g\phi} & \Big\downarrow g \\
 & & k^n
\end{array} .$$

Now, this is so because change of basis on k^r already is taken care of in $\mathrm{Grass}(r, n)$. Notice that this is the reason for the action of

$$\mathrm{Sl}(r_1) \times \cdots \times \mathrm{Sl}(r_N) \times \mathrm{Sl}(n)$$

on

$$\mathrm{Hom}(k^{r_1}, k^n) \oplus \cdots \mathrm{Hom}(k^{r_N}, k^n)$$

on page 211 in GIT[2], and just the action of $\mathrm{Sl}(n+1)$ on $\mathrm{Grass}(r, n+1)$ on page 86 in GIT[2].

6 The N-subspace problem

Consider Grass $/r, n)$. This is the set of $n \times r$-matrices of rank r, i.e. maximal rank, and can be considered as the open subset of $\mathbb{A}^{rn}$ consisting of matrices of rank r. Let $s_{i_1,\ldots,i_{n-r}}$ be the cofactor determinant of a $n \times r$-matrix, and put $Z = Z(\bigcup_{i_1,\ldots,i_{n-r}} s_{i_1,\ldots,i_{n-r}})$. then

$$\mathrm{Grass}(r, n) = (\mathbb{A}^{nr} - Z)/\,\mathrm{Gl}(r)$$

in analogy with $\mathbb{P}^n = \mathrm{Grass}(1, n+1) = (\mathbb{A}^{n+1} - \{0\})/\,\mathrm{Gl}(1)$. We push the analogy further: Consider the affine cone over $\mathrm{Grass}(r, n)$, and let $\mathrm{Gl}(n)$ act equivariant:

$$\mathrm{Grass}(r, n) \twoheadleftarrow \mathbb{A}^{nr} - Z \hookrightarrow \mathbb{A}^{nr}$$

A point $x \in \text{Grass}(r,n)$ is stable if $o(\hat{x})$ is closed in $\mathbb{A}^{nr}$ and $\dim o(\hat{x}) = \dim G$ for one and hence all points $\hat{x}$ over x. It is semistable if $\overline{o(\hat{x})} \bigcap Z = \emptyset$. Otherwise (i.e. $\overline{o(\hat{x})} \bigcap Z \neq \emptyset$) the point x is called unstable. Notice that the plücker embedding of $\text{Grass}(r,n)$ in $\mathbb{P}^{s-1}$, $s = \binom{n}{r}$, gives the setup

$$\widehat{\text{Grass}(r,n)} \subseteq \qquad \mathbb{A}^s$$

$$\widehat{\text{Grass}(r,n)} - \{0\} \subseteq \qquad \mathbb{A}^s - \{0\}$$

$$\text{Grass}(r,n) \subseteq \qquad \mathbb{P}^{s-1}$$

Then the situation is no longer intrinsic but depends upon a polarization. To work on the N-subspace problem, we study

$$\text{Grass}(r_1,n) \times \cdots \times \text{Grass}(r_N,n) = G(r_1,\ldots;r_N)$$

by lifting the $\text{Gl}(n)$-action to

$$\mathbb{A}^{r_1 n} \times \cdots \times \mathbb{A}^{r_1 N}.$$

Notice that the following computations are justified by lemma 4 because all the modules in question have suport strictly inside an open affine subset. This is easy to see, but have to be considered in each separate case.

As is usual, we start with an example.

6.1 $G(1,1;2)$.

Let us first consider only the $\text{Gl}(2)$-action: Then we consider $\mathbb{A}^2 \times \mathbb{A}^2 \cong \mathbb{A}^4$. We write the elements in $\mathbb{A}^4$ on the form $v \in \mathbb{A}^4$, $v = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)$, and $g \in \text{Gl}(2)$ acts by $gv = \left(g \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad g \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = g \cdot \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)$. There are two orbits in $\text{Grass}(1,2)$ in this case. The case where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ are paralell, that is $a = b$, and the case where they are not. The orbit $o(a,a)$ is contained in $Z(x_1 - y_1, x_2 - y_2)$ which obviously is the closure of this orbit. So (a,a) is indeed an unstable point. The above orbit can be seen as $o\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. The the other point can be seen as $o\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is obvious that

$$\text{cl}(o\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}) = Z(x_1 - y_1, x_2 - y_2)),$$

and because $o\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{A}^4 - Z(\text{Det})$ is an open dense subset, it follows that

$$\text{cl}(o\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \mathbb{A}^4.$$

In particular, the point (a, b) is also an unstable point. Thus the situation we consider is

$$A = k[x_1, x_2, y_1, y_2],\ V_1 = A,\ V_2 = A/(x_1 - y_1, x_2 - y_2).$$

Given these objects, to construct the non commutative scheme, we start by computing the non commutative prorepresenting hull. As we know, this starts with a computation of the tangent spaces. Notice that as $V_1 = A$ is free,

$$\text{Ext}^1_{A-G}(V_1, V_1) \subseteq \text{Ext}^1_A(V_1, V_1) = 0.$$

Also

$$\text{Ext}^1_{A-G}(V_1, V_2) = 0$$

of the same reason. We have that

$$\text{Ext}^1_{A-G}(V_2, V_1) = 0$$

by writing up the syzygies, and we are left with $\text{Ext}^1_{A-G}(V_2, V_2)$ as the only thing to compute. $\text{Ext}^1_{A-G}(V_2, V_2)$ is given by the invariants ϕ in the diagram

$$(x_1 - y_1, x_2 - y_2) \xrightarrow{\nabla_g} (x_1 - y_1, x_2 - y_2) \xrightarrow{\phi} A/(x_1 - y_1, x_2 - y_2) \xrightarrow{\nabla_{g^{-1}}} A/(x_1 - y_1, x_2 - y_2)$$

If $\phi = (f_1, f_2)$ is invariant for $g = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ we must have $f_1 = ax_1,\ f_2 = bx_2$ for $a, b \in k$. Then if $\phi = (ax_1, bx_2)$ is invariant under $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$(ax_1, bx_2) = \nabla_g \phi \nabla_{g^{-1}} = (bx_2, ax_1)\nabla_{g^{-1}} = (bx_1, ax_2)$$

implying that $a = b$. Thus $\phi = \alpha(x_1, x_2)$. We have to check that ϕ is invariant under $g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $g^{-1} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$:

$$\nabla_g(x_1, x_2)\nabla_{g^{-1}} = (x_1, x_2 + cx_1)\nabla_{g^{-1}} = (x_1, x_2 + cx_1 - cx_1) = (x_1, x_2).$$

Thus $\text{ext}^1_{A-G}(V_2, V_2) = 1$. The Koszul complex gives

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & V_2 & \longleftarrow & A & \longleftarrow & A^2 & \xleftarrow{\begin{pmatrix} x_2 - y_2 \\ y_1 - x_1 \end{pmatrix}} & A & \longleftarrow & 0 \\ & & & & & \overset{\alpha}{\swarrow} & & \overset{\beta}{\swarrow} & & & \\ 0 & \longleftarrow & V_2 & \longleftarrow & A & \underset{(x_1 - y_1, x_2 - y_2)}{\longleftarrow} & A^2 & \longleftarrow & A & \longleftarrow & 0 \end{array}$$

where $\alpha = (x_1, x_2)$ and $\beta = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$. We compute that the cup-products are immediately zero, and so

$$H = \begin{pmatrix} k & 0 \\ 0 & k[t] \end{pmatrix}.$$

Notice that This is not G(1, 1; 2). Up to both actions, all pairs of non parallel lines are equivalent, and all "pairs" of parallel lines are also. Thus there is no free parameter. We have to take both actions into account simultaneously.

Two lines in $\mathbb{A}^2$ is a 2×2-matrix $M = \begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix}$ where both columns have rank 1. The group $\mathrm{Gl}(1) \times \mathrm{Gl}(1) \times \mathrm{Gl}(2)$ acts on M, and we end up with two possible orbits, $o(M_1)$ and $o(M_2)$ where:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We find that $o(M_1) = D(\mathrm{Det}\, M) \Rightarrow \mathrm{cl}(o(M_1)) = \mathbb{A}^4$, and that $o(M_2) = Z(\mathrm{Det})$. This situation is the following:

$$A = k[x_{11}, y_{12}, x_{21}, x_{21}, y_{22}],\ V_1 = A,\ V_2 = A/s,\ s = \mathrm{Det}\,.$$

By writing up the syzygies it is obvious that

$$\mathrm{Ext}^1_A(V_1, V_1) = \mathrm{Ext}^1_A(V_1, V_2) = \mathrm{Ext}^1_A(V_2, V_1) = 0,$$

and so also

$$\mathrm{Ext}^1_{A-G}(V_1, V_1) = \mathrm{Ext}^1_{A-G}(V_1, V_2) = \mathrm{Ext}^1_{A-G}(V_2, V_1) = 0.$$

We consider The sequence

$$s \xrightarrow{\nabla_g} s \xrightarrow{\phi} A/s \xrightarrow{\nabla_{g^{-1}}} A/s\ .$$

Letting $g_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ and $\phi(s) = f$ we find that the only possibility is $f = s = 0$ Thus also $\mathrm{Ext}^1_{A-G}(V_2, V_2) = 0$, and so

$$H(1, 1; 2) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

6.2 G(2,2;3)

The previous example was of course too simple. This is also. Here we consider $\mathrm{Grass}(2, 3) \times \mathrm{Grass}(2, 3)$. That is, we consider the open subset of all 3×4-matrices consisting of those on the form $M = (V_1|V_2)$ where $V_i,\ i = 1, 2$ is a

3×2-matrix of rank 2. The group action on $\mathrm{Hom}(k^2, k^3) \times \mathrm{Hom}(k^2, k^3)$ that we will consider is given by

$$G = \mathrm{Gl}(2) \times \mathrm{Gl}(2) \times \mathrm{Gl}(3),$$

$\nabla_{(g_1,g_2,g_3)}(V_1|V_2) = g_3 \cdot (V_1 \cdot g_1 | V_2 \cdot g_2)$.

Notice that this is not the Grassmanian, but a quotient of this.

Lemma 13. *Every element in* $\mathrm{Grass}(2,3) \times \mathrm{Grass}(2,3)$ *is equivalent to (is in the orbit of) a matrix in one of the forms*

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. We start by proving that every element is in the orbit of one of the following matrices:

$$M_{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; M_E = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\mathrm{Grass}(2,3)$ we have the following possibilities:

$$\begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{pmatrix}, \; \begin{pmatrix} 1 & a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Combining these to get $G(2,2;3)$ we get nine possibilities. We will only prove one of the possibilities, and leave the rest.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ a_1 & a_2 & b_1 & b_2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & b_1 - a_1 & b_2 - a_2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & c_1 & c_2 \end{pmatrix} = M(c_1, c_2)$$

i)

$c_1 = c_2 = 0 \Rightarrow M(0,0) = M_E$

ii)

$$c_1 \neq 0, c_2 = 0 \Rightarrow M(c_1, 0) \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\infty.$$

iii)

$$c_1 = 0, c_2 \neq 0 \Rightarrow M(0, c_2) \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$M_{(0)}$

iv)

$c_1 \neq 0, c_2 \neq 0 \Rightarrow M(c_1,c_2) \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{c_1}{c_2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{c_1}{c_2} \end{pmatrix} \sim$
$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = M_\infty.$

Finally, we use the fact that there exists a linear transformation that sends the two planes M_λ to M_∞. Then we are through. □

As before, we consider $G = \mathrm{Gl}(2) \times \mathrm{Gl}(2) \times \mathrm{Gl}(3)$ as the algebraic group $\mathrm{Spec}[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}] = \mathrm{Spec}([\underline{x}, \underline{y}])$ acting on

$$\mathbb{A}(M(3,2)) \times \mathbb{A}(M(3,2)) = \mathbb{A}^{12}.$$

We have two orbits, the orbit of two non parallel planes, and the orbit of two identical planes. Let $o_1 = o(V_1)$, $o_\infty = o(V_\infty)$ where

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, V_\infty = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The closure of these two orbits is described algebraically by the following:

Let $\begin{pmatrix} x_{11} & x_{12} & y_{11} & y_{12} \\ x_{21} & x_{22} & y_{21} & y_{22} \\ x_{31} & x_{32} & y_{31} & y_{32} \end{pmatrix}$, s_i, $i = 1, \dots, 4$ be the determinant of the matrix resulting from removing the i'th column from M. The matrices in the orbit of V_1 are given by the fact that the rank is 3, i.e.

$$o(V_1) = \bigcup_{i=1}^{4} D(s_i) \subseteq \mathbb{A}^{12}.$$

Because $\mathbb{A}^{12}$ is irreducible, it follows that $\mathrm{cl}(o(V_1)) = \mathbb{A}^{12}$. Also, the orbit of V_2 is given by the fact that the rank is 2 so that

$$o(V_2) \subseteq \bigcap_{i=1}^{4} Z(s_i) = Z(s_1, s_2, s_3, s_4).$$

Now, because $o(V_2) = \bigcup_{i,j,k}(D(s_{ijk}))$ is open, it follows as above that $\mathrm{cl}(o(V_2)) = Z(s_1, s_2, s_3, s_4) \subseteq \mathbb{A}^{12}$. This leads us to the following classification problem:

$$A = k[\underline{x}, \underline{y}], \ M_1 = A, \ M_2 = A/(s_1, \dots, s_4).$$

$\mathrm{Ext}^1_{A-G}(M_2, M_2)$

The action is given by the composition

$$(s_1, s_2, s_3, s_4) \xrightarrow{\nabla_g} (s_1, s_2, s_3, s_4) \xrightarrow{\phi} A/(s_1, s_2, s_3, s_4) \xrightarrow{\nabla_{g^{-1}}} A/(s_1, s_2, s_3, s_4).$$

Let $\phi = (f_1, f_2, f_3, f_4)$, $f_i \in A$, $i = 1, \ldots, 4$. Let $g = (\mathrm{Id}, \mathrm{Id}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})$

Then
$g \cdot \phi = \nabla_g \circ \phi \circ \nabla_{g^{-1}} = (\alpha f_i(\frac{1}{\alpha}x_{11}, \frac{1}{\alpha}x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, \frac{1}{\alpha}y_{11}, \frac{1}{\alpha}y_{12}, y_{21}, y_{22}, y_{31}, y_{32}))$.

Because ϕ should also be invariant for $g = (\mathrm{Id}, Id, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix})$, it follows that $\phi = 0$.

$\mathrm{Ext}^1_{A-G}(M_2, M_1)$

From general principles, we know that this should be zero. Also it follows by writing up the resolution of M_2. All other $Ext^1_{A-G}(M_i, M_j)$ is obviously zero, and so we get

$$H(2, 2; 3) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

6.3 $G(2, 2; 4)$

The last example $G(2, 2; 3)$ is equivalent to two lines in affine 3-space, and is as such well known. The last example proved that $G(2, 2; 4)$ is a commutative scheme, but it remains an open question if this is the case in general.

Going one step further, we are going to compute the quotient

$$\mathrm{Grass}(2, 4) \times \mathrm{Grass}(2, 4) / \mathrm{Gl}(4),$$

and we will do it the following way: Consider an element $(V_1, V_2) \in \mathrm{Grass}(2, 4) \times \mathrm{Grass}(2, 4)$ as an equivalens class of two linear morphisms of rank 2. That is two 4×2-matrices, each of rank 2, modulo $\mathrm{Gl}(2) \times \mathrm{Gl}(2)$. Consider $\mathbb{A}^{16}$ as the scheme of all 4×4 matrices. Let U be the open set consisting of all such matrices (V_1, V_2) where $\mathrm{rk}\, V_i = 2$, $i = 1, 2$. That is the following: Let

$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}.$$

Let f_{ij} be the determinant of the matrix coming from M by removing the third and fourth columns and the i'th and j'th rows, correspondingly for g_{ij}, the determinant of the matrix coming from M by removing the first and second columns and the i'th and j'th rows. Then $U \subseteq \mathbb{A}^{16}$ consisting of the matrices corresponding to 2 linear subspaces of dimension 2 is

$$U = (\bigcup_{i \neq j} D(f_{ij})) \bigcap (\bigcup_{i \neq j} D(g_{ij}))$$

which certainly is open in $\mathbb{A}^{16}$.

Now, what we are going to compute, is

$$U/G$$

where $G = \mathrm{Gl}(2) \times \mathrm{Gl}(2) \times \mathrm{Gl}(4)$, the action given by

$$(h_1, h_2, g) \cdot (V_1, V_2) = g(V_1 h_1, V_2 h_2).$$

U is not affine, and so we classify the closure of the orbits in $\mathbb{A}^{16}$. This will work properly because two different orbits have different closures.

Lemma 14. *The orbits in U under the action of G are the orbits of one of the following matrices*

$$o_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; o_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; o_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

that is, two planes intersecting in the origin only, two planes with a common line and finally, two identical planes.

Proof. Follows by choosing the bases the suitable way □

The orbit of o_1 consists of all matrices with rank 4, that is all matrices M with $\det(M) \neq 0$. Thus $\mathrm{cl}(o_1) = \mathrm{cl}(D(\det(M))) = \mathbb{A}^{16}$. The orbit of o_2 consists of every element in U of rank 3. Its closure is $Z(\det(M))$. Finally, the closure of o_3 is contained in $Z(s_{ij})$, and so $\mathrm{cl}(o_3) = Z(s_{ij})$, where s_{ij} is the ij cofactor of M. The $A-G$-modules we are going to classify are

$$V_1 = A, \; V_2 = A/(s), \; V_3 = A/(s_{ij})$$

where $A = k[x_{ij}]$, $s = \det(M)$, $s_{ij} = ij$-cofactor of M.

As always, the next step in the construction of the (not necessarily commutative) local formal moduli, is the computation of the tangent space. This leads to some combinatorial difficulties, as the computation below will show. First notice that because G is reductive, when $\mathfrak{p} \subseteq \mathfrak{q}$ are g-invariant ideals, then

$$\mathrm{Ext}^1_{A-G}(A/\mathfrak{p}, A/\mathfrak{q}) = \mathrm{Ext}^1_{A-G}(A/\mathfrak{p}, A/\mathfrak{q})^G = \mathrm{Hom}(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})^G.$$

We also would like to recall that the action of $g \in G$ on $\phi \in \mathrm{Hom}(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})$ is given by the composition

$$\mathfrak{p} \xrightarrow{\nabla_g} \mathfrak{p} \xrightarrow{\phi} A/\mathfrak{q} \xrightarrow{\nabla_{g^{-1}}} A/\mathfrak{q}.$$

Then we get the following computation.

$\mathrm{Ext}^1_{A-G}(V_2, V_2)$. Consider the composition

$$(s) \xrightarrow{\nabla_g} (s) \xrightarrow{\phi} A/(s) \xrightarrow{\nabla_{g^{-1}}} A/(s),$$

and notice that ∇_g is homogeneous for all $g \in G$, and that all ideals in question are homogeneous. Then we can work homogeneous. Moreover, ϕ is determined by its value on$s$, thus we may write

$$\phi = \sum_{k=0}^{n} f^k$$

were f^k is the homogeneous part of degree k. Assume that ϕ is invariant. Then it is certainly invariant under $g = (\mathrm{Id}, \mathrm{Id}, \alpha \cdot \mathrm{Id})$, thus

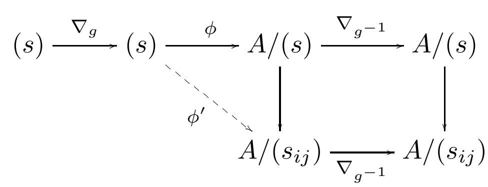
$$\phi = \nabla_g \circ \phi \circ \nabla_{g^{-1}} \Leftrightarrow \sum_{k=0}^{n} f^k = \sum_{k=0}^{n} \alpha^4 \frac{1}{\alpha^k} f^k = \sum_{k=0}^{n} \alpha^{4-k} f^k.$$

This implies that $f^k = 0$ for $\neq 4$, and we may write $\phi = f^4$, that is a polynomial of degree 4.

Choosing a monomial basis for the monomials of degree 4, we understand that all monomials that are not elementary products must be 0. Two entries from the same row or column would destroy the invariance of multiplication of that row or column with a constant $\alpha \neq 0, 1$. Then, invariance under switching of rows leads to $f^4 = \alpha \cdot \det(M)$, because then we are running through all elementary products, changing signs, That is

$$\mathrm{ext}^1_{A-G}(V_2, V_2) = 0.$$

$\mathrm{Ext}^1_{A-G}(V_2, V_3)$. Because of the diagram



it follows that $0 \leq \mathrm{ext}^1_{A-G}(V_2, V_3) \leq \mathrm{ext}^1_{A-G}(V_2, V_2) = 0 \Rightarrow$

$$\mathrm{ext}^1_{A-G}(V_2, V_3) = 0$$

$\mathrm{Ext}^1_{A-G}(V_3, V_3)$. We consider the composition

$$(s_{ij}) \xrightarrow{\nabla_g} (s_{ij}) \xrightarrow{\phi} A/(s_{ij}) \xrightarrow{\nabla_{g^{-1}}} A/(s_{ij}).$$

We can write $\phi = (f_{ij}) = \sum\limits_{k=0}^{n}(f_{ij}^k)$, where f_{ij}^k are homogeneous of degree k. Assume that ϕ is invariant under the action of G,i.e. under the composition above. Then in particular it is invariant under $g = (\text{Id}, \text{Id}, \alpha \cdot \text{Id})$, that is

$$\phi = \nabla_g \circ \phi \circ \nabla_{g^{-1}} \Leftrightarrow \sum_{k=0}^{n}(f_{ij}^k) = \sum_{k=0}^{n}(\alpha^3 \frac{1}{\alpha^k} f_{ij}^k) = \sum_{k=0}^{n}(\alpha^{3-k} f_{ij}^k).$$

This implies that $f_{ij}^k = 0$ for $k \neq 3$, and we can write $\phi = (f_{ij}^3)$ where each f_{ij}^3 is homogeneous of degree 3. For the same reason as above, the f_{ij}^3's can not contain two entries from the same row or column. Thus they are elementary products. Switching rows now runs through all elementary products changing signs, and so $f_{ij} = \sum\limits_{i,j}\alpha_{ij}s_{ij}$. This says

$$\text{ext}^1_{A-G}(V_3, V_3) = 0.$$

It is obvious that $\text{Ext}^1_{A-G}(V_1, V_j) = \text{Ext}^1_{A-G}(A, V_j) = 0$ for all j. Thus our result is that the moduli problem $G(2, 2; 4)$ is commutative:

$$H(2, 2; 4) = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

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