Solving random walk recursions in the plane using separation of variables and Fourier analysis

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1 Introduction

It is well known that many mathematical problems, originating in "real world", may be modeled by an infinitesimal approach, resulting in some kind of differential equation. The method of doing local considerations also applies in the discrete case, such as random walks. In that case it is often called a one step analysis, and the result will be a recursion equation instead of a differential equation. The aim of this article is first of all to show how to solve a two-dimensional random walk problem on an arbitrary rectangle with certain boundary conditions, using separation of variables and discrete Fourier analysis. Second, my intention is to demonstrate how ordinary Fourier analysis will solve the same problem if residue calculations are applied. Finally, I pay special attention to a resulting formula, connecting discrete and ordinary Fourier coefficients.

2 One-dimensional random walk

There are a lot of different aspects to one-dimensional random walk. Here we will consider random walk on the non-negative numbers, starting at the number m, $1 \leq m \leq M-1$. A successful walk will end at position M, while a walk that ends at position 0, is considered a failure. The walker moves stepwise to the neighbouring integer on the right hand side with probability

$p > 0$ and to the left with probability $q = 1 - p$. Let $u(m)$ be the probability that the random walk reaches the target number M, given the initial position m.

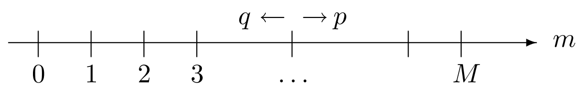

Figure 1 Random walk on the integers

In textbooks, this problem is often solved, making use of the corresponding recursion equation, appearing from a one-step analysis (se for example [1])

$$u(m) = p \cdot u(m+1) + q \cdot u(m-1)$$

Assuming solutions $u(m) = r^m$, the characteristic equation $pr^2 - r + q = 0$ is generated. Combining the solutions $r_1 = 1$ and $r_2 = \frac{q}{p}$, and taking the boundary conditions $u(0) = 0$ og $u(M) = 1$ into consideration, we obtain the solution

$$u(m) = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^M} & \text{when } p \neq q \\ \frac{m}{M} \quad \text{when } p = q = \frac{1}{2} \end{cases}$$

Another context to this problem is of course the "gambler's ruin", a game between two players, A and B, attempting to win the total amount of money, say M dollars. Every time they play a partial game, they only risk one dollar each. The initial fortune of player A is assumed to be m, so that player B initially has a fortune $M - m$. Let $u(m)$ be the probability that player A ends up with all the money (and B is ruined). To make this situation fit into the model, let p be the probability that A wins an arbitrary partial game.

B's winning probability is therefore $q = 1 - p$. If, for instance, $p = 0.45$ ($q = 0.55$) and $M = 1000$, the formula implies that A's initial fortune must be at least $m = 997$ dollars to make A's total winning probability exceed 50%. It is quite surprising to observe that the solution of a second degree equation is nearly all we need to solve a relatively complicated probabilistic problem of random walk.

In general I believe that teachers of mathematics should bear in mind that probability theory historically originated from gambling problems, and that there is reason to believe that gambling contexts still motivate students in a powerful way.

3 Two-dimensional random walk

3.1 Describing the situation

We will now consider a random walk in the plane, within the rectangle $[1, M-1] \times [1, N-1]$, where $M \geq 2$ and $N \geq 2$ are integers, as shown in figure 2.

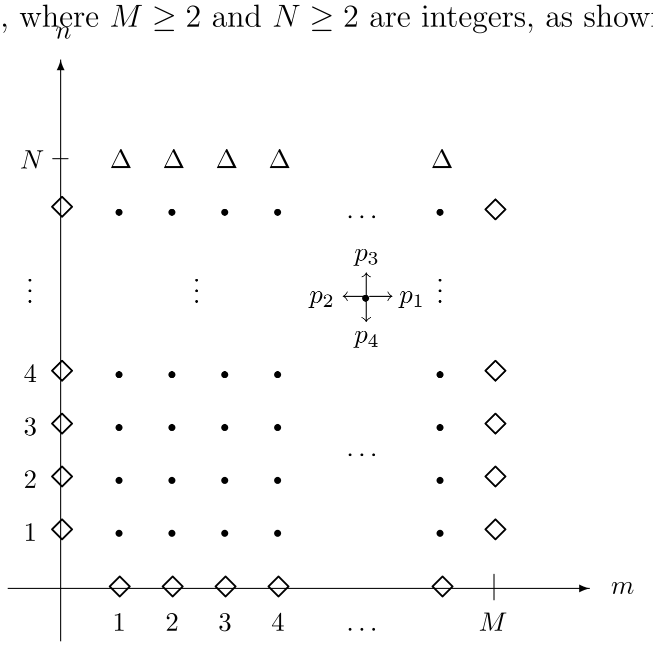

Figure 2 Random walk in the plane

We assume that the walk starts at an inner point (m, n). The walker moves to the neighbouring points $(m+1, n)$, $(m-1, n)$, $(m, n+1)$ og $(m, n-1)$ with probabilities p_1, p_2, p_3 and p_4, respectively. Assume that $\sum_i p_i = 1$ and $0 < p_i < 1, i = 1..4$. Assume furthermore that the walk is successful when ending at the upper horizontal part of the boundary (symbol Δ), and likewise the walk is considered a failure if some other boundary point (symbol $\diamond$) is visited first.

Let $u(m, n)$ be the probability that the random walk is successful, starting at (m, n). Then u will satisfy the following partial difference equation:

$$u(m, n) = p_1 \cdot u(m+1, n) + p_2 \cdot u(m-1, n) + p_3 \cdot u(m, n+1) + p_4 \cdot u(m, n-1) \quad (1)$$

Obviously we have the boundary conditions:

$$u(0, n) = u(M, n) = 0, \quad n = 1..N-1$$
$$u(m, 0) = 0 \text{ and } u(m, N) = 1, \quad m = 1..M-1$$

This problem is essentially solved in the symmetrical case by [2], but without the aid of Fourier analysis. [3] uses matrix formalism to solve this and other related problems. Random walks on other kinds of lattices than square lattices have been treated by for example [4] and [5], using separation of variables as an important tool.

3.2 Linear system of equations

One can easily realize that our problem in principle may be handled by solving a linear system of equations. We have $(M-1)(N-1)$ unknowns, $u(m, n)$ ($m = 1..M-1$ og $n = 1..N-1$), and the same number of equations. The standard argument that there exists a unique solution (see [6]) is that $u(m, n)$ becomes a weighed average of the u-values of the neighbouring points and therefore cannot be smaller than all of them. We conclude that both maximum and minimum must be obtained at the boundary. An inhomogeneous system (like this one) with the same number of unknowns and equations, has a unique solution if the corresponding homogeneous system is only solved trivially by $u(m,n) \equiv 0$. If $u(m,n) = 0$ on the whole boundary, then consequently $u(m,n) = 0$ for all inner points too. So the solution is unique. Since the boundary values are either 0 or 1, the $u(m,n)$-values must lie between 0 and 1 inside the rectangle, which is required for probabilities. For the simple case $M = N = 4$ we get nine equations and nine unknowns. Assuming $p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$, we obtain the solutions in the Table 1.

$n \backslash m$	1	2	3
3	$\frac{3}{7}$	$\frac{59}{112}$	$\frac{3}{7}$
2	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$
1	$\frac{1}{14}$	$\frac{11}{112}$	$\frac{1}{14}$

Table 1 We observe expected symmetry in the solutions.

3.3 Separation of variables

In analogy to a standard partial differential equation approach, separation of variables requires that solutions of (1) fulfil:

$$u(m,n) = F(m)G(n) \tag{2}$$

In addition we want the three homogeneous boundary conditions $u(0,n) = u(M,n) = 0$ og $u(m,0) = 0$ to be satisfied.
Putting (2) into (1), we find

$$\frac{p_1 F(m+1) + p_2 F(m-1)}{F(m)} = 1 - \frac{p_3 G(n+1) + p_4 G(n-1)}{G(n)} \tag{3}$$

Since the left hand side of (3) only depends on m and the right hand side only depends on n, and since F and G are independent, it follows that both sides must be a common constant, called 2λ (practical reasons for the factor 2). Out of this we get two recursion relations

$$p_1 F(m+1) - 2\lambda F(m) + p_2 F(m-1) = 0 \quad \text{where } F(0) = F(M) = 0 \qquad (4)$$

and

$$p_3 G(n) + (2\lambda - 1)G(n+1) + p_4 G(n-1) = 0 \quad \text{where } G(0) = 0 \qquad (5)$$

First we consider equation (4). Trivial solutions are avoided by demanding complex solutions of the corresponding characteristic equation

$$p_1 \alpha^2 - 2\lambda\alpha + p_2 = 0$$

We solve and find

$$\alpha_{1,2} = \frac{\lambda \pm i\sqrt{p_1 p_2 - \lambda^2}}{p_1} \quad \text{where } \alpha_1 \alpha_2 = \frac{p_2}{p_1} \quad \text{and } |\lambda| < \sqrt{p_1 p_2}$$

giving

$$F(m) = A\alpha_1^m + B\bar{\alpha}_1^m$$

where $\alpha_1 = \frac{\lambda + i\sqrt{p_1 p_2 - \lambda^2}}{p_1}$.

$F(0) = 0$ gives $B = -A$ and $F(M) = 0$ gives

$$\alpha_1^M - \bar{\alpha}_1^M = 0$$

or, by multiplying α_1^M to each side,

$$\alpha_1^{2M} = \left(\frac{p_2}{p_1}\right)^M$$

This leads to

$$\alpha_1 = \frac{\lambda + i\sqrt{p_1 p_2 - \lambda^2}}{p_1} = \sqrt{\frac{p_2}{p_1}} \left(\cos\left(\frac{k\pi}{2M}\right) + i \sin\left(\frac{k\pi}{2M}\right)\right)$$

Now we can identify λ

$$\lambda = \lambda_k = \sqrt{p_1 p_2} \cos\left(\frac{k\pi}{M}\right)$$

Consequently we have

$$F(m) = F_k(m) = A \cdot \left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \left\{e^{\frac{k\pi i m}{M}} - e^{-\frac{k\pi i m}{M}}\right\} = \tilde{A}\left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \sin\left(\frac{k\pi m}{M}\right)$$

Applying this λ_k-expression in equation (5), we get a new corresponding characteristic equation

$$p_3\beta^2 + (2\lambda_k - 1)\beta + p_4 = 0 \tag{6}$$

We solve and find

$$\beta_{1,2}(k) = \frac{1 - 2\lambda_k \pm \sqrt{(1-2\lambda_k)^2 - 4p_3p_4}}{2p_3} \quad \text{where } \beta_1\beta_2 = \frac{p_4}{p_3}$$

Since $|\lambda_k| < \sqrt{p_1p_2} \leq \frac{1}{2}$, we have

$$(1-2\lambda_k)^2 > (1 - 2\sqrt{p_1p_2})^2$$

Define $f = \left(1 - 2\sqrt{p_1p_2}\right)^2 - 4p_3p_4$, $\bar{p} = \frac{1}{2}(p_3+p_4)$ and $s = |\bar{p}-p_3| = |\bar{p}-p_4|$

Then we have $4p_3p_4 = 4(\bar{p}-s)(\bar{p}+s) = 4\bar{p}^2 - 4s^2 = 4\bar{p}^2 - (p_3-p_4)^2$. It is now possible to rewrite f, such that

$$f = (1-2\sqrt{p_1p_2})^2 - 4\bar{p}^2 + (p_3-p_4)^2 = (1-2\sqrt{p_1p_2}-2\bar{p})(1-2\sqrt{p_1p_2}+2\bar{p}) + (p_3-p_4)^2$$

If we keep in mind that $2\bar{p} = p_3 + p_4 = 1 - p_1 - p_2$, we can rewrite f once more to get

$$f = (1 - 2\sqrt{p_1p_2} - (1-p_1-p_2))(1 - 2\sqrt{p_1p_2} + (1-p_1-p_2)) + (p_3-p_4)^2$$

Now it is easily seen that

$$f = (\sqrt{p_1} - \sqrt{p_2})^2\{2 - (\sqrt{p_1}+\sqrt{p_2})^2\} + (p_3-p_4)^2$$

which means that $f \geq 0$.

This implies that the characteristic equation (6) has two separate and real solutions, β_1 and β_2. Let $\beta_1 > \beta_2$. We find

$$G(n) = C\beta_1^n(k) + D\beta_2^n(k)$$

The condition $G(0) = 0$ gives us $D = -C$, so that

$$G(n) = C\left(\beta_1^n(k) - \beta_2^n(k)\right)$$

Therefore, the relevant solutions of the partial difference equation, satisfying the three homogeneous boundary conditions, are given by

$$u_k(m,n) = B_k\left(\beta_1^n(k) - \beta_2^n(k)\right)\left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \sin\left(\frac{k\pi m}{M}\right)$$

3.4 Discrete Fourier analysis

To satisfy the inhomogeneous boundary conditions $u(m,N) = 1$, $m = 1 \ldots M-1$, we have to make use of linear combinations of the u_k-functions. In ordinary Fourier analysis we would try an infinite series, but the fact that $\sin\left(\frac{k\pi m}{M}\right)$ is periodic in k, will make such an approach a detour, yet an interesting one (see section 3.5). Therefore, let us first try

$$u(m,n) = \sum_{k=1}^{M-1} B_k\left(\beta_1^n(k) - \beta_2^n(k)\right)\left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \sin\left(\frac{k\pi m}{M}\right) \tag{7}$$

The conditions $u(m,N) = 1$, $m = 1..M-1$, lead to

$$\sum_{k=1}^{M-1} B_k\left(\beta_1^N(k) - \beta_2^N(k)\right)\sin\left(\frac{k\pi m}{M}\right) = \left(\frac{p_1}{p_2}\right)^{\frac{m}{2}}, m = 1,\ldots,M-1$$

for appropriately chosen B_k. To those familiar with discrete Fourier transforms, it is clear that the inhomogeneous conditions will be satisfied if we choose B_k such that

$$B_k\left(\beta_1^N(k) - \beta_2^N(k)\right) = c_k, k = 1,\ldots,M-1 \tag{8}$$

where c_k is closely related to the discrete Fourier transform of $\left(\frac{p_1}{p_2}\right)^{\frac{m}{2}}$, extended as an odd function, $m = -M + 1..M - 1$. Then we get (see for eksemple [7])

$$c_k = \frac{2}{M}\sum_{j=1}^{M-1}\left(\frac{p_1}{p_2}\right)^{\frac{j}{2}}\sin\left(\frac{k\pi j}{M}\right) = \frac{2}{M}\Im\left(\sum_{j=1}^{M-1}\left(\frac{p_1}{p_2}\right)^{\frac{j}{2}}e^{\frac{k\pi j}{M}i}\right)$$

We realize that this is a finite geometric series, giving

$$c_k = \frac{2}{M}\Im\left(\frac{1-\left(\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}e^{\frac{k\pi i}{M}}\right)^M}{1-\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}e^{\frac{k\pi i}{M}}}\right) = \frac{2}{M}\frac{\left(1-\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}}\cos(k\pi)\right)\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\sin\left(\frac{k\pi}{M}\right)}{1-2\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\cos\left(\frac{k\pi}{M}\right)+\frac{p_1}{p_2}}$$

With the aid of (8) we therefore have

$$B_k = \frac{2}{M}\frac{\left(1-\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}}\cos(k\pi)\right)\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\sin\left(\frac{k\pi}{M}\right)}{\left(1-2\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\cos\left(\frac{k\pi}{M}\right)+\frac{p_1}{p_2}\right)\left(\beta_1^N(k)-\beta_2^N(k)\right)}$$

Choosing B_k according to this and with

$$\beta_{1,2}(k) = \frac{1-2\lambda_k \pm \sqrt{(1-2\lambda_k)^2-4p_3p_4}}{2p_3} \quad \text{where} \quad \lambda_k = \sqrt{p_1p_2}\cos\left(\frac{k\pi}{M}\right),$$

$u(m, n)$ from (7) will solve the given random walk problem.

3.5 Ordinary Fourier analysis

As mentioned in the previous section, it should be possible, but certainly no shortcut, to apply ordinary Fourier analysis to get the inhomogeneous boundary condition satisfied. We then try

$$u(m,n) = \sum_{k=0}^{\infty} u_k(m,n) = \sum_{k=1}^{\infty} u_k(m,n)$$

In this setting we give B_k the new name b_k. The conditions $u(m,N) = 1$, $m = 1..M-1$, imply

$$\sum_{k=1}^{\infty} b_k \left(\beta_1^N(k) - \beta_2^N(k)\right) \sin\left(\frac{k\pi m}{M}\right) = \left(\frac{p_1}{p_2}\right)^{\frac{m}{2}}, m = 1, \ldots, M-1$$

Ordinary Fourier analysis, considering the odd extension of $\left(\frac{p_1}{p_2}\right)^{\frac{x}{2}}$ to the interval $[-M, M]$, renders the following formula:

$$b_k = \frac{2\left(\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}} + 1 - 2\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}} \cos^2\left(\frac{k\pi}{2}\right)\right) k\pi}{\left(\frac{1}{4}\ln^2\left(\frac{p1}{p2}\right) M^2 + k^2\pi^2\right)\left(\beta_1^N - \beta_2^N\right)}$$

Choosing b_k according to this will give the nice solution

$$u(m,n) = \sum_{k=1}^{\infty} b_k \cdot \left(\beta_1^n(k) - \beta_2^n(k)\right) \left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \sin\left(\frac{k\pi m}{M}\right), m \in [-M, M] \quad (9)$$

3.6 Reaching the finite Fourier series from the infinite

Let us see how it is possible to deduce the finite Fourier series solution (7) from the infinite series (9). We rewrite (9) as follows:

$$u(m,n) = \sum_{k=1}^{\infty} \frac{k}{a^2 + b^2k^2} \gamma(k) \quad (10)$$

where

$$\gamma(k) = \frac{2\pi\left(\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}} + 1 - 2\left(\frac{p_1}{p_2}\right)^{\frac{M}{2}} \cos^2\left(\frac{k\pi}{2}\right)\right)\left(\beta_1^n(k) - \beta_2^n(k)\right)\left(\frac{p_2}{p_1}\right)^{\frac{m}{2}} \sin\left(\frac{k\pi m}{M}\right)}{\beta_1^N(k) - \beta_2^N(k)} \quad (11)$$

and

$$a = \frac{1}{2}\ln\left(\frac{p_1}{p_2}\right) M, b = \pi \quad (12)$$

It is easy to see that when $k = 1..M-1$ and $q = 0, 1, \ldots$, we have

$$\gamma(k) = \gamma(k + 2qM) = -\gamma(2M - k + 2qM) \quad (13)$$

It is also true that $\gamma(rM) = 0$ when r is a nonnegative integer. This is why we get

$$u(m,n) = \sum_{k=1}^{M-1} \gamma(k) \sum_{q=0}^{\infty} \left(\frac{k+2qM}{a^2 + b^2(k+2qM)^2} - \frac{2M-k+2qM}{a^2 + b^2(2M-k+2qM)^2} \right) \tag{14}$$

and furthermore

$$u(m,n) = \sum_{k=1}^{M-1} \gamma(k) \sum_{q=0}^{\infty} 2\delta(q,k) \tag{15}$$

where we define $\delta(q,k)$:

$$\delta(q,k) = \frac{(k-M)(b^2k^2 - 2b^2Mk + a^2 - 4b^2q^2M^2 - 4b^2qM^2)}{(a^2 + b^2(k+2qM)^2)(a^2 + b^2(2M-k+2qM)^2)} \tag{16}$$

The fact that $\delta(0,k) = \delta(-1,k)$, $\delta(1,k) = \delta(-2,k)$, ..., enables us to express $u(m,n)$ as the sum

$$u(m,n) = \sum_{k=1}^{M-1} \gamma(k) \sum_{q=-\infty}^{\infty} \delta(q,k) \tag{17}$$

The denominator of $\delta(q,k)$ has simple poles in $q_{1,2} = -\frac{k}{2M} \pm \frac{a}{2Mb}i$ and $q_{3,4} = \frac{k}{2M} - 1 \pm \frac{a}{2Mb}i$, so by standard residue calculus (see for eksemple [8]) we have

$$\sum_{q=-\infty}^{\infty} \delta(q,k) = -\pi \sum_{i=1}^{4} Res_{q=q_i} \left(\cot(\pi q)\delta(q,k)\right) = -\pi \sum_{i=1}^{4} \cot(\pi q) \lim_{q \to q_i} (q-q_i)\delta(q,k)$$

Using (16), we find (here I was assisted by *Maple*):

$$\sum_{q=-\infty}^{\infty} \delta(q,k) = \frac{\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} \sin\left(\frac{k\pi}{M}\right)}{\pi M \left(1 - 2\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} \cos\left(\frac{k\pi}{M}\right) + \frac{p_1}{p_2}\right)} \tag{18}$$

Combining this formula with (11) and (17), we find that the resulting solution $u(m,n)$ coincides with the one found by discrete Fourier analysis, given by (7).

3.7 Comparing with Laplace's differential equation

Laplace's two-dimensional equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{19}$$

Let the two partial difference operators Δ_1 og Δ_2 be defined by

$$\Delta_1 u(m,n) = u(m+1,n) - u(m,n)$$

and

$$\Delta_2 u(m,n) = u(m,n+1) - u(m,n)$$

Then we will have

$$\Delta_1^2 u(m-1,n) = u(m+1,n) - 2u(m,n) + u(m-1,n)$$

and

$$\Delta_2^2 u(m,n-1) = u(m,n+1) - 2u(m,n) + u(m,n-1)$$

Laplace's differential equation in discrete form is therefore given by

$$\Delta_1^2 u(m-1,n) + \Delta_2^2 u(m,n-1) = 0$$

og equivalently

$$4u(m,n) = u(m+1,n) + u(m-1,n) + u(m,n+1) + u(m,n-1)$$

which is the symmetric recursion relation treated in this article.

Let us consider (19) with boundary conditions

$$u(0,y) = u(M,y) = u(x,0) = 0 \quad \text{og} \quad u(x,N) = 1$$

This becomes a standard Dirichlet problem with solution (see for example [9])

$$u(x,y) = \sum_{l=1}^{\infty} \frac{4\sin\left(\frac{(2l-1)\pi x}{M}\right)\sinh\left(\frac{(2l-1)\pi y}{M}\right)}{(2l-1)\pi\sinh\left(\frac{(2l-1)\pi N}{M}\right)} \tag{20}$$

If we compare $u(x,y)$ from (20) to $u(m,n)$, given in (9), with $p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$ and $k = 2l-1$, we observe several similarities:

$$u(m,n) = \sum_{l=1}^{\infty} \frac{4}{(2l-1)\pi} \cdot \frac{\beta_1^n - \beta_2^n}{\beta_1^N - \beta_2^N} \cdot \sin\left(\frac{(2l-1)\pi m}{M}\right)$$

We have $\beta_1 = a + \sqrt{a^2-1}$ og $\beta_2 = a - \sqrt{a^2-1}$, where $a = 2 - \cos r$ and $r = \frac{(2l-1)\pi}{M}$. Further on we remark that $\beta_1\beta_2 = 1$, so that

$$\beta_1^n - \beta_2^n = e^{n\ln(a+\sqrt{a^2-1})} - e^{n\ln(a-\sqrt{a^2-1})} = 2\sinh(n\ln(a+\sqrt{a^2-1}))$$

Consequently we have

$$\frac{\beta_1^n - \beta_2^n}{\beta_1^N - \beta_2^N} = \frac{\sinh(n\ln(a+\sqrt{a^2-1}))}{\sinh(N\ln(a+\sqrt{a^2-1}))}$$

If $r = \frac{(2l-1)\pi}{M}$ becomes small, we can make use of the Taylor expansion $\ln(2-\cos r + \sqrt{(2-\cos r)^2-1}) \sim r$, leading to

$$\frac{\beta_1^n - \beta_2^n}{\beta_1^N - \beta_2^N} \approx \frac{\sinh(nr)}{\sinh(Nr)} = \frac{\sinh(\frac{(2l-1)\pi n}{M})}{\sinh(\frac{(2l-1)\pi N}{M})}$$

When l is getting big, we get small contributions to the series

$$u(m,n) = \sum_{l=1}^{\infty} \frac{4}{(2l-1)\pi} \cdot \frac{\beta_1^n - \beta_2^n}{\beta_1^N - \beta_2^N} \cdot \sin\left(\frac{(2l-1)\pi m}{M}\right)$$

because of the factor $\frac{4}{(2l-1)\pi}$ being small. This implies that little harm can be done, using the Taylor expansion even when r is big.

The numerical deviations between the discrete and the continuous case turned out to be small even for relatively small M and N. The case $M = N = 10$, is demonstrated in table 2 and table 3. Table 2 gives the probabilities $u(m,n)$, based on the discrete model while Table 3 similarly gives $u(x,y)$, based on the continuous model.

$n\backslash m$	1	3	5	7	9
9	0.489	0.754	0.799	0.754	0.489
7	0.179	0.402	0.466	0.402	0.179
5	0.083	0.207	0.250	0.207	0.083
3	0.038	0.098	0.120	0.098	0.038
1	0.011	0.029	0.035	0.029	0.011

Table 2

$y\backslash x$	1	3	5	7	9
9	0.489	0.759	0.802	0.759	0.489
7	0.175	0.403	0.468	0.403	0.175
5	0.082	0.206	0.250	0.206	0.082
3	0.038	0.097	0.119	0.097	0.038
1	0.011	0.029	0.035	0.029	0.011

Table 3

4 Fourier coefficients - a connecting formula

Let f be a function, continuous and piecewise smooth on $[0, M]$. Assume in addition $f(0) = 0$. A discrete Fourier representation of $f(m)$ ($m = 1 \ldots M -$

1) is then given by

$$f(m) = \sum_{k=1}^{M-1} c_k \sin\left(\frac{k\pi m}{M}\right), \quad c_k = \frac{2}{M}\sum_{m=1}^{M-1} f(m)\sin\left(\frac{k\pi m}{M}\right)$$

In a similar manner we get

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{M}\right), \quad b_k = \frac{2}{M}\int_0^M f(x)\sin\left(\frac{k\pi x}{M}\right)\, dx$$

The special case $x = m$:

$$f(m) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi m}{M}\right) = \sum_{k=1}^{M-1} \sin\left(\frac{k\pi m}{M}\right) \cdot \sum_{q=0}^{\infty} \delta(q,k)$$

where $\delta(q,k) = b_{k+2qM} - b_{2M-k+2qM}$.

Combining the two expressions of $f(m)$ will now give us (because of linear independence)

$$c_k = \sum_{q=0}^{\infty} \delta(q,k)$$

If the intention is to make the right hand side more computable in the sense of enabling us to apply residue calculus (as in section 3.6), we first observe that $b_k = -b_{-k}$, so that, for nonnegative q, we get

$$\delta(-q-1,k) = b_{-2M+k-2qM} - b_{-k-2qM} = -b_{2M-k+2qM} + b_{k+2qM} = \delta(q,k)$$

The alternative formula of c_k is consequently given by

$$c_k = \frac{1}{2}\sum_{q=-\infty}^{\infty} \delta(q,k) = \frac{1}{2}\sum_{q=-\infty}^{\infty} b_{k+2qM} - b_{2M-k+2qM}$$

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